

EXTREMES OF GAUSSIAN RANDOM FIELDS WITH REGULARLY VARYING DEPENDENCE STRUCTURE

KRZYSZTOF DĘBICKI, ENKELEJD HASHORVA, AND PENG LIU

Abstract: Let $X(t), t \in \mathcal{T}$ be a centered Gaussian random field with variance function $\sigma^2(\cdot)$ that attains its maximum at the unique point $t_0 \in \mathcal{T}$, and let $M(\mathcal{T}) := \sup_{t \in \mathcal{T}} X(t)$. For \mathcal{T} a compact subset of \mathbb{R} , the current literature explains the asymptotic tail behaviour of $M(\mathcal{T})$ under some regularity conditions including that $1 - \sigma(t)$ has a polynomial decrease to 0 as $t \rightarrow t_0$. In this contribution we consider more general case that $1 - \sigma(t)$ is regularly varying at t_0 . We extend our analysis to random fields defined on some compact $\mathcal{T} \subset \mathbb{R}^2$, deriving the exact tail asymptotics of $M(\mathcal{T})$ for the class of Gaussian random fields with variance and correlation functions being regularly varying at t_0 . A crucial novel element is the analysis of families of Gaussian random fields that do not possess locally additive dependence structures, which leads to qualitatively new types of asymptotics.

Key Words: Non-stationary Gaussian processes; Gaussian random fields; extremes; fractional Brownian motion; regular variation; uniform approximation

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1. INTRODUCTION

Let $X(t), t \geq 0$ be a centered stationary Gaussian processes with continuous trajectories, unit variance and correlation function $r(\cdot)$ satisfying Pickands's condition

$$(1) \quad 1 - r(t) \sim a|t|^\alpha, \quad t \downarrow 0, \quad a > 0, \quad \text{and } r(t) < 1, \forall t \neq 0,$$

with $\alpha \in (0, 2]$; in our notation \sim means asymptotic equivalence when the argument tends to 0 or ∞ .

In the seminal contribution [26] it was shown that under (1), for any T positive

$$(2) \quad \mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim T \mathcal{H}_\alpha a^{1/\alpha} u^{2/\alpha} \mathbb{P} \{X(0) > u\}, \quad u \rightarrow \infty,$$

with the classical Pickands constant \mathcal{H}_α defined by

$$\mathcal{H}_\alpha = \lim_{T \rightarrow \infty} T^{-1} \mathbb{E} \left\{ \sup_{t \in [0, T]} e^{\sqrt{2} B_\alpha(t) - t^\alpha} \right\},$$

where $B_\alpha(t), t \geq 0$ is a standard fractional Brownian motion with Hurst index $\alpha/2 \in (0, 1]$, see [26, 27, 28, 9, 16, 8, 17, 10, 29, 11, 15, 6] for various properties of \mathcal{H}_α .

The above finding was extended in various directions, including $\alpha(t)$ -locally-stationary Gaussian processes (see [14]), and general non-stationary Gaussian processes and random fields, see e.g., [29]. A particularly important place in this theory is taken by the result of Piterbarg and Prisjažnjuk [30], where the exact tail asymptotics of $\sup_{t \in [0, T]} X(t)$ is derived in the case that the variance function σ^2 of a centered Gaussian process X has a unique point of maximum in $[0, T]$, say t_0 . More precisely, for the correlation function it is assumed therein that for some $\alpha \in (0, 2]$

$$(3) \quad 1 - r(s, t) \sim a|t - s|^\alpha, \quad s, t \downarrow 0, \quad a > 0,$$

whereas the behaviour of the variance function around the unique maximizer t_0 of $\sigma^2(t)$ over $[0, T]$ such that $\sigma(t_0) = 1$, is supposed to satisfy

$$(4) \quad 1 - \sigma(t) \sim b|t|^\beta, \quad t \downarrow 0, \quad b > 0, \quad \beta \in (0, \infty).$$

Assume further that the following Hölder continuity condition

$$(5) \quad \mathbb{E} \{ (X(t) - X(s))^2 \} \leq C|t - s|^\nu$$

is valid for all $s, t \in [0, \theta]$ with some $\theta \in (0, T]$ and $\nu \in (0, 2]$, by [30], for $\alpha < \beta$

$$(6) \quad \mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim \mathcal{H}_\alpha \frac{a^{1/\alpha}}{b^{1/\beta}} \Gamma(1/\beta + 1) u^{2/\alpha - 2/\beta} \mathbb{P} \{ X(0) > u \},$$

and for $\alpha = \beta$

$$(7) \quad \mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim \mathcal{P}_\alpha^{b/a} \mathbb{P} \{ X(0) > u \},$$

where $\mathcal{P}_\alpha^d, d > 0$ is the Piterbarg constant defined by

$$\mathcal{P}_\alpha^d = \lim_{S \rightarrow \infty} \mathbb{E} \left\{ \sup_{t \in [0, S]} e^{\sqrt{2}B_\alpha(t) - (1+d)t^\alpha} \right\} \in (0, \infty).$$

When $\alpha > \beta$, then (7) holds with 1 instead of $\mathcal{P}_\alpha^{b/a}$; see also Theorem 2.1 in [10] for the case $T = \infty$.

We note in passing that in fact the Hölder continuity (5) is not needed to derive the asymptotics of (2), which will be shown later in our main theorems; necessary and sufficient conditions that guarantee the global Hölder continuity of X are presented in the deep contribution [1].

The original Pickands assumption (1), and its counterpart (3) can be relaxed to $1 - r$ being regularly varying at 0 with index $\alpha \in (0, 2]$, see [33, 2]. Specifically, in the case of a non-stationary X we shall assume for some non-negative $\rho \in \mathcal{R}_{\alpha/2}, \alpha \in (0, 2]$

$$(8) \quad 1 - r(s, t) \sim \rho^2(|t - s|), \quad s, t \downarrow 0.$$

Here $f \in \mathcal{R}_\gamma$ means that the function f is regularly varying at 0 with index γ , see [34, 19, 36] for details.

The first goal of this contribution is an extension of Piterbarg's results to a more general setup, that is to suppose that

$$(9) \quad 1 - \sigma(t) \sim v^2(t), \quad t \downarrow 0,$$

where $v \geq 0$ and $v \in \mathcal{R}_{\beta/2}, \beta > 0$. In Theorem 2.1 we show that the asymptotic tail behaviour of $\sup_{t \in [0, T]} X(t)$ can be determined under the assumption that $1 - \sigma$ can be compared with $1 - \rho$, namely if further

$$(10) \quad \lim_{t \downarrow 0} \frac{v^2(t)}{\rho^2(t)} = \gamma \in [0, \infty].$$

Note that, in Piterbarg's result mentioned above the limit γ is assumed to exist.

Then we analyze tail distribution asymptotics of supremum of centered Gaussian random field $X(s, t), s \in [-T_1, T_1], t \in [-T_2, T_2]$ with unique point that maximizes its variance function, say $(0, 0)$. Although extremes of Gaussian random fields with regularly varying correlation function are discussed in [33], see also [4, 5, 18, 25, 31, 13, 22, 32] for new developments on extremes of Gaussian random fields, most of the results in the existing literature are focused on the analysis of fields with *locally additive* dependence structure, that is if

$$\text{Var}(X(0, 0)) - \text{Var}(X(s, t)) \sim A_1|s|^{\beta_1} + A_1|t|^{\beta_2}$$

and

$$1 - \text{Corr}(X(s, t), X(s_1, t_1)) \sim B_1|s - s_1|^{\alpha_1} + B_2|t - t_1|^{\alpha_2}$$

as $s, s_1 \rightarrow 0, t, t_1 \rightarrow 0$. It appears that the investigation of fields that do not satisfy this properties is considerably more delicate and leads to qualitatively new results. In Section 3 we derive several novel results concerned with the exact tail asymptotics of the maximum of centered Gaussian random fields when both variance and correlation functions are

regularly varying and do not possess a locally additive structure.

Brief outline of the rest of the paper: Our main result for extremes of Gaussian processes is displayed in the Section 2, whereas Section 3 covers Gaussian random fields. The proofs of the theorems are presented in Section 4 and some technical results and their proofs are relegated to Appendix A and B.

2. GAUSSIAN PROCESSES

Before continuing with our investigation, we mention first that there are indeed important cases of Gaussian processes that satisfy our general setup in Section 1. Indeed, as remarked in [23] and [24], for any function $\rho^2 \in \mathcal{R}_\alpha, \alpha \in (0, 2]$ there exists a centered stationary Gaussian process Y with continuous trajectories, unit variance and correlation function r satisfying (8). Clearly, for any continuous function $\sigma(t), t \geq 0$ the process $X(t) = \sigma(t)Y(t), t \geq 0$ has continuous trajectories and variance function σ^2 .

One instance for the properties of σ is to assume that (9) holds with

$$v^2(t) = |\ln t|^c t^\beta, \quad A > 0, c \in \mathbb{R}, \beta > 0.$$

For such σ , only the case $c = 0$ can be dealt with using Piterbarg's result mentioned in the Introduction. It is tempting to write

$$v^2(t) = (|\ln t|^{c/\beta} t)^\beta.$$

Since in Piterbarg's result condition (4) explains the asymptotic expansion in (6) the $u^{-2/\beta}$ term when $\alpha < \beta$, the above could imply that (6) still holds if we replace $u^{-2/\beta}$ by $|\ln u|^{-2c/\beta^2} u^{-2/\beta}$.

Detailed calculations show that this intuition does not lead to the correct result, and in fact the problem is much more complicated. Indeed, the tail asymptotics of the supremum is determined in this case in terms of the (unique) asymptotic inverse of v , which is given by (see Example 1.24 in [36] or Lemma 2 in [21])

$$\overleftarrow{v}(t) \sim \left(\frac{\beta}{2}\right)^{c/\beta} |\ln t|^{-c/\beta} t^{2/\beta}, \quad t \downarrow 0,$$

where \overleftarrow{f} denotes the asymptotic (unique) inverse of $f \in \mathcal{R}_\gamma$.

Hereafter all regularly varying functions at 0 are assumed to be ultimately non-negative as $t \rightarrow 0$. Further $\Psi(u) \sim e^{-u^2/2}/(\sqrt{2\pi}u)$, as $u \rightarrow \infty$, denotes the tail distribution of an $N(0, 1)$ random variable, and we set

$$\mathcal{P}_\alpha^\infty =: 1, \quad \mathcal{P}_\alpha^\infty[0, S] =: 1, \quad \alpha \in (0, 2], S > 0.$$

We state next the main result of this section.

Theorem 2.1. *Let $X(t), t \geq 0$ be a centered Gaussian process with continuous trajectories and variance function σ^2 having unique maximum at 0 with $\sigma(0) = 1$. Suppose that σ satisfies (9) and the correlation function r of X satisfies (8). Assume further that condition (10) is valid for some $\gamma \in [0, \infty]$.*

i) If $\gamma = 0$, then

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim \Gamma(1/\beta + 1) \mathcal{H}_\alpha \frac{\overleftarrow{v}(1/u)}{\overleftarrow{\rho}(1/u)} \Psi(u).$$

ii) If $\gamma \in (0, \infty]$, then

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim \mathcal{P}_\alpha^\gamma \Psi(u).$$

Remarks 2.2. *i) If the maximum point of the variance is not 0, but an inner point, say $t_0 \in (0, T)$ such that $\sigma(t_0) = 1$, then the results of Theorem 2.1 remain valid with \mathcal{H}_α replaced by $\widehat{\mathcal{H}}_\alpha$ and $\mathcal{P}_\alpha^\gamma$ replaced by $\widehat{\mathcal{P}}_\alpha^\gamma$, where*

$$\widehat{\mathcal{H}}_\alpha = 2\mathcal{H}_\alpha, \quad \widehat{\mathcal{P}}_\alpha^\gamma = \lim_{S \rightarrow \infty} \mathbb{E} \left\{ \sup_{t \in [-S, S]} e^{\sqrt{2}B_\alpha(t) - (1+\gamma)t^\alpha} \right\}, \quad \widehat{\mathcal{P}}_\alpha^\infty = 1.$$

ii) Since Theorem 2.1 remain valid if we substitute v by an asymptotically equivalent v^* , we can assume that $v^2(t) = \ell_\sigma(t)t^\beta$ with ℓ_σ a normalized slowly varying function (see e.g., [3, 36]). Similarly, let $\rho^2(t) = \ell_\rho(t)t^\alpha$ with ℓ_ρ another normalized slowly varying function. Set next

$$\ell_{\rho,\alpha}(x) = \sqrt{\ell_\rho(x^{1/\alpha})}, \quad \ell_{\sigma,\beta}(x) = \sqrt{\ell_\sigma(x^{1/\beta})}.$$

If further $\ell_{\sigma,\beta}^\#$ and $\ell_{\rho,\alpha}^\#$ denote the asymptotic inverses of $\ell_{\sigma,\beta}$ and $\ell_{\rho,\alpha}$, respectively then we have

$$v(x) = \ell_{\sigma,\beta}(x^\beta)x^{\beta/2}, \quad \rho(x) = \ell_{\rho,\alpha}(x^\alpha)x^{\alpha/2}$$

and thus by Example 1.24 in [36] as $t \rightarrow 0$

$$\overleftarrow{v}(t) \sim [\ell_{\sigma,\beta}^\#(t)]^{2/\beta}t^{2/\beta}, \quad \overleftarrow{\rho}(t) \sim [\ell_{\rho,\beta}^\#(t)]^{2/\alpha}t^{2/\alpha}.$$

Consequently,

$$\frac{\overleftarrow{v}(1/u)}{\overleftarrow{\rho}(1/u)} \sim u^{2/\alpha-2/\beta} \frac{[\ell_{\sigma,\beta}^\#(1/u)]^{2/\beta}}{[\ell_{\rho,\beta}^\#(1/u)]^{2/\alpha}}, \quad u \rightarrow \infty.$$

Theorem 2.1 is useful also when dealing with additive Gaussian random field. Specifically, assume that for $T_1, T_2 > 0$

$$X(s, t) = \eta_1(s) + \eta_2(t), \quad s \in [-T_1, T_1], t \in [-T_2, T_2],$$

with η_1, η_2 two independent centered Gaussian random processes with continuous trajectories. If both η_1 and η_2 are stationary satisfying (1), or η_1 and η_2 satisfy the conditions of Theorem 2.1, then

$$\mathbb{P} \left\{ \sup_{t \in [-T_i, T_i]} \eta_i(t) > u \right\} \sim \mathcal{L}_i(u) u^{\tau_i} e^{-u^2/2}$$

for some $\tau_i \geq -1$ with $\mathcal{L}_i(x) = C, x \geq 0$ if $\tau_i = -1$ and \mathcal{L}_i is slowly varying at infinity if $\tau_i > -1$. Hence, since

$$\sup_{s \in [-T_1, T_1], t \in [-T_2, T_2]} X(s, t) = \sup_{s \in [-T_1, T_1]} \eta_1(s) + \sup_{t \in [-T_2, T_2]} \eta_2(t),$$

then Lemma 2.3 in [20] implies

$$(11) \quad \mathbb{P} \left\{ \sup_{s \in [-T_1, T_1], t \in [-T_2, T_2]} X(s, t) > u \right\} \sim \sqrt{2\pi} \mathcal{L}_1(u) \mathcal{L}_2(u) u^{\tau_1+\tau_2-1} e^{-u^2/4}, \quad u \rightarrow \infty.$$

In the particular case that η_i 's satisfy the conditions of Theorem 2.1 with $\rho_i, v_i, i = 1, 2$ instead of ρ and v , where

$$(12) \quad \lim_{t \downarrow 0} \frac{v_i^2(t)}{\rho_i^2(t)} = \gamma_i \in [0, \infty], i = 1, 2,$$

then (11) can be given more explicitly, see Theorem 3.1 below.

As we show in the next section, general Gaussian random fields are much more complex to deal with, and the results cannot be derived from Theorem 2.1.

3. GAUSSIAN RANDOM FIELDS

Extremes of locally additive Gaussian random fields with regularly varying correlation function are discussed in [33]. However there are no results in the literature if the variance function is determined in terms of regularly varying functions and the dependence structure is non additive. In order to motivate our study, we consider first the additive Gaussian random field $X(s, t) = \eta_1(s) + \eta_2(t), s \in [-T_1, T_1], t \in [-T_2, T_2]$ introduced in Section 2. Thus, using that the variance function $\sigma^2(s, t)$ of $X(s, t)$ is simply given by

$$\sigma^2(s, t) = \sigma_1^2(s) + \sigma_2^2(t),$$

if η_1, η_2 satisfy the assumptions of Theorem 2.1, then $\sigma(s, t)$ achieves its unique maximum at $(0, 0)$.

In this section we shall discuss an extension of Theorem 2.1 to

$$(13) \quad \mathbb{P} \left\{ \sup_{(s,t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\},$$

as $u \rightarrow \infty$, where $X(s, t), (s, t) \in [-T_1, T_1] \times [-T_2, T_2]$ is a centered Gaussian random field, with variance function that is maximal on a unique point but possess dependence structure that is more complex than the additive one discussed above. In particular, we suppose that

$$(14) \quad 1 - r(s, t, s_1, t_1) \sim \rho_1^2(|a_{11}(s - s_1) + a_{12}(t - t_1)|) + \rho_2^2(|a_{21}(s - s_1) + a_{22}(t - t_1)|)$$

as $s, s_1, t, t_1 \rightarrow 0$ with $\rho_i \geq 0$ and $\rho_i \in \mathcal{R}_{\alpha_i/2}, \alpha_i \in (0, 2], i = 1, 2$.

For the variance function $\sigma^2(s, t) = \text{Var}(X(s, t))$ we shall assume that it attains its maximum at the unique point $(0, 0)$ with $\sigma(0, 0) = 1$ and further

$$(15) \quad 1 - \sigma(s, t) \sim v_1^2(|b_{11}(s - s_1) + b_{12}(t - t_1)|) + v_2^2(|b_{21}(s - s_1) + b_{22}(t - t_1)|), \quad s, t \downarrow 0,$$

where $v_i \geq 0$ and $v_i \in \mathcal{R}_{\beta_i/2}, \beta_i > 0, i = 1, 2$.

We note that recent results for the case that the variance function σ^2 is maximal on a line, which is the case for instance if η_1 is stationary with unit variance 1 and η_2 satisfies the assumptions of Theorem 2.1, are obtained in [7].

For further analysis it is useful to introduce the following matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Let us observe that the assumption of uniqueness of the maximizer of $\sigma(\cdot, \cdot)$ implies that $\text{rank}(B) = 2$.

We shall assume that (12) holds and furthermore the following limits

$$(16) \quad \lim_{t \downarrow 0} \frac{\rho_2^2(t)}{\rho_1^2(t)} = \eta \in [0, \infty], \quad \lim_{t \downarrow 0} \frac{v_2^2(t)}{v_1^2(t)} = \theta \in [0, \infty]$$

exist.

It appears that the rank of matrix A plays the key role for the asymptotics of (13), as $u \rightarrow \infty$. Thus, in what follows, we shall distinguish between two scenarios, when $\text{rank}(A) = 2$ and $\text{rank}(A) = 1$. We exclude from further analysis the degenerated case of $\text{rank}(A) = 0$.

3.1. Scenario I: $\text{rank}(A) = 2$. Suppose that A is invertible and observe that $Y(s, t) := X((A^{-1}(s, t)^\top)^\top)$ has, under (14), (15), correlation function such that

$$(17) \quad 1 - r_Y(s, s_1, t, t_1) \sim \rho_1^2(|s - s_1|) + \rho_2^2(|t - t_1|), \quad s, s_1, t, t_1 \rightarrow 0,$$

and variance function σ_Y^2 satisfying

$$(18) \quad 1 - \sigma_Y(s, t) \sim v_1^2(|c_{11}s + c_{12}t|) + v_2^2(|c_{21}s + c_{22}t|), \quad s, t \rightarrow 0,$$

with

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = BA^{-1}.$$

Therefore, with no loss of generality, in this section we tacitly assume that X satisfies (14) with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =: I.$$

Next, define an additive fractional Brownian field W by

$$W(s, t) = \sqrt{2}B_\alpha(s) + \sqrt{2}\tilde{B}_\alpha(t) - |s|^\alpha - |t|^\alpha,$$

where $B_\alpha(t)$ and $\tilde{B}_\alpha(t)$ are independent standard fBm's with index $\alpha \in (0, 2]$. For a given matrix $D = (d_{ij})_{i,j=1,2}$, we define the generalized Piterbarg constant

$$\hat{\mathcal{P}}_\alpha^{\gamma_1, \gamma_2, D} := \lim_{S \rightarrow \infty} \mathbb{E} \left\{ \sup_{(s, t) \in [-S, S]^2} e^{W(s, t) - \gamma_1 |d_{11}s + d_{12}t|^\alpha - \gamma_2 |d_{21}s + d_{22}t|^\alpha} \right\},$$

where $\gamma_1, \gamma_2 > 0$. Note that if $\det(D) \neq 0$, then there exists $\gamma_3 > 0$ such that

$$\gamma_1 |d_{11}s + d_{12}t|^\alpha + \gamma_2 |d_{21}s + d_{22}t|^\alpha \geq \gamma_3 (|s|^\alpha + |t|^\alpha), \quad s, t \in \mathbb{R},$$

which implies that $\widehat{\mathcal{P}}_{\alpha}^{\gamma_1, \gamma_2, D} \leq \left(\widehat{\mathcal{P}}_{\alpha}^{\gamma_3}\right)^2 < \infty$. Moreover, for $D = I$ we have

$$\widehat{\mathcal{P}}_{\alpha}^{\gamma_1, \gamma_2, I} = \widehat{\mathcal{P}}_{\alpha}^{\gamma_1} \widehat{\mathcal{P}}_{\alpha}^{\gamma_2}.$$

Let for S_1, S_2 non-negative

$$\mathcal{H}_{\alpha}^{\gamma_1, \gamma_2, b}(S_1, S_2) := \mathbb{E} \left\{ \sup_{(s+bt, t) \in [-S_1, S_1] \times [0, S_2]} e^{W(s, t) - \gamma_1 |s+bt|^{\alpha} - \gamma_2 |t|^{\alpha}} \right\},$$

and

$$\widehat{\mathcal{H}}_{\alpha}^{\gamma_1, \gamma_2, b}(S_1, S_2) := \mathbb{E} \left\{ \sup_{(s+bt, t) \in [-S_1, S_1] \times [-S_2, S_2]} e^{W(s, t) - \gamma_1 |s+bt|^{\alpha} - \gamma_2 |t|^{\alpha}} \right\}.$$

In order to simplify the notation we set

$$\mathcal{H}_{\alpha}^{\gamma_1, \gamma_2, b}(S_1) := \mathcal{H}_{\alpha}^{\gamma_1, \gamma_2, b}(S_1, S_1), \quad \widehat{\mathcal{H}}_{\alpha}^{\gamma_1, \gamma_2, b}(S_1) = \widehat{\mathcal{H}}_{\alpha}^{\gamma_1, \gamma_2, b}(S_1, S_1), \quad \mathcal{H}_{\alpha}^{\gamma_1, b}(S_1) = \mathcal{H}_{\alpha}^{\gamma_1, 0, b}(S_1, S_1),$$

and

$$\mathcal{H}_{\alpha}^{\gamma_1, b} := \lim_{S \rightarrow \infty} S^{-1} \mathcal{H}_{\alpha}^{\gamma_1, b}(S).$$

Now let us proceed to the analysis of (13) for four special cases whose proofs are all different, and to which one can reduce all other scenarios (as will be advocated at the end of this section).

Since below A is taken to be the identity matrix, the cases discussed below are defined by the different choices of the matrix B .

◇ Case 1. We say that X is *locally additive*, if both (14) and (15) hold with $A = B = I$. The result below holds for any $\theta, \eta \in [0, \infty]$ defined in (16).

Theorem 3.1. *Suppose that X is a locally additive Gaussian random field.*

i) If $\gamma_1 = \gamma_2 = 0$, then

$$\mathbb{P} \left\{ \sup_{(s, t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim 4 \prod_{i=1}^2 \left(\Gamma(1/\beta_i + 1) \mathcal{H}_{\alpha_i}^{\frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}} \right) \Psi(u).$$

ii) If $\gamma_1 = 0, \gamma_2 \in (0, \infty]$, then

$$\mathbb{P} \left\{ \sup_{(s, t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim 2\Gamma(1/\beta_1 + 1) \mathcal{H}_{\alpha_1} \widehat{\mathcal{P}}_{\alpha_2}^{\gamma_2} \frac{\overleftarrow{v}_1(1/u)}{\overleftarrow{\rho}_1(1/u)} \Psi(u).$$

iii) If $\gamma_1, \gamma_2 \in (0, \infty]$, then

$$\mathbb{P} \left\{ \sup_{(s, t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim \widehat{\mathcal{P}}_{\alpha_1}^{\gamma_1} \widehat{\mathcal{P}}_{\alpha_2}^{\gamma_2} \Psi(u).$$

Remark 3.2. *We note that by the use of change of coordinates Theorem 3.1 covers all the combinations of values of γ_1, γ_2 .*

◇ Case 2. Here we shall assume that (14) and (15) are satisfied with

$$(19) \quad A = I, \quad B = \begin{pmatrix} 1 & b_{12} \\ 0 & 1 \end{pmatrix}, \quad \text{with } b_{12} \neq 0.$$

Theorem 3.3. *Suppose that (16) is satisfied with $\eta \in (0, \infty)$, $\theta = 0$ and (19) holds.*

i) If $\gamma_1 = \gamma_2 = 0$, then

$$\mathbb{P} \left\{ \sup_{(s, t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim 4 \prod_{i=1}^2 \left(\Gamma(1/\beta_i + 1) \mathcal{H}_{\alpha_i}^{\frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}} \right) \Psi(u).$$

ii) If $\gamma_2 = 0, \gamma_1 \in (0, \infty]$, then

$$\mathbb{P} \left\{ \sup_{(s, t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim 2\Gamma(1/\beta_1 + 1) \mathcal{H}_{\alpha_1}^{\gamma_1, b_{12}\eta^{-1/\alpha_1}} \frac{\overleftarrow{v}_1(1/u)}{\overleftarrow{\rho}_1(1/u)} \Psi(u).$$

iii) If $\gamma_2 \in (0, \infty]$, $\gamma_1 = \infty$, then

$$\mathbb{P} \left\{ \sup_{(s,t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim \widehat{\mathcal{P}}_{\alpha_1}^{\gamma_2(|b_{12}|^{\alpha_1} \eta^{-1} + 1)^{-1}} \Psi(u).$$

Remark 3.4. The above theorem covers all the possible combinations of values of γ_1, γ_2 , since the assumption that $\eta \in (0, \infty)$, $\theta = 0$ excludes cases $\gamma_1 = 0, \gamma_2 \in (0, \infty]$ and $\gamma_1 \in (0, \infty), \gamma_2 \in (0, \infty]$.

Although the same asymptotics are imposed in i) of Theorem 3.1 and i) of Theorem 3.3, their proofs require a substantially different approach. Thus we did not combine those cases in one result.

◇ **Case 3.** The assumptions on A and B are the same as in Case 2 above, however we shall suppose that $\eta = 0$, $\theta \in (0, \infty)$. Since $\theta \in (0, \infty)$, we set $\beta = \beta_1 = \beta_2$. Let $\mu \in (-\infty, \infty)$ be the point at which $|1 + b_{12}t|^\beta + \theta|t|^\beta$ attains its minimum over $(-\infty, \infty)$. We have $\mu \in [-1/|b_{12}|, 1/|b_{12}|]$. Further, set

$$(20) \quad M_\beta = \inf_{t \in (-\infty, \infty)} (|1 + b_{12}t|^\beta + \theta|t|^\beta)$$

and define the two-sided Piterbarg-type constant

$$\widehat{\mathcal{P}}_\beta^{g_s} = \lim_{S \rightarrow \infty} \widehat{\mathcal{P}}_\beta^{g_s}[-S, S], \quad \text{with} \quad \widehat{\mathcal{P}}_\beta^{g_s}[-S, S] = \mathbb{E} \left\{ \sup_{t \in [-S, S]} e^{\sqrt{2}B_\beta(t) - t^\beta - g_s(t)} \right\}, \quad S > 0, s \geq 0,$$

where

$$g_s(t) = \theta^{-1} \gamma_2 (|s + b_{12}t|^\beta + \theta|t|^\beta - |(1 + b_{12}\mu)s|^\beta - \theta|\mu s|^\beta), \quad s \geq 0, t \in \mathbb{R}.$$

Further, set

$$\mathcal{I}_\beta := \int_{-\infty}^{\infty} \widehat{\mathcal{P}}_\beta^{g_{|s|}} e^{-\frac{\gamma_2 M_\beta}{\theta} |s|^\beta} ds \in (0, \infty).$$

The finiteness of \mathcal{I}_β follows from the fact that for any $\epsilon > 0$, there exists a positive constant $c_\epsilon > 0$ such that

$$g_s(t) + \epsilon|s|^\beta \geq c_\epsilon|t|^\beta, \quad s \geq 0, t \in \mathbb{R}$$

implying that $\widehat{\mathcal{P}}_\beta^{g_s} \leq \widehat{\mathcal{P}}_\beta^{c_\epsilon} e^{\epsilon s^\beta} < \infty$, and thus for $\epsilon \in (0, \theta^{-1} \gamma_2 M_\beta)$

$$\mathcal{I}_\beta \leq 2 \int_0^\infty \widehat{\mathcal{P}}_\beta^{g_s} e^{-\frac{\gamma_2 M_\beta}{\theta} s^\beta} ds \leq 2 \widehat{\mathcal{P}}_\beta^{c_\epsilon} \int_0^\infty e^{-(\frac{\gamma_2 M_\beta}{\theta} - \epsilon) s^\beta} ds < \infty.$$

Theorem 3.5. Suppose that (19) holds and (16) is satisfied with $\eta = 0$, $\theta \in (0, \infty)$.

i) If $\gamma_1 = \gamma_2 = 0$, then

$$\mathbb{P} \left\{ \sup_{(s,t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim 4 \prod_{i=1}^2 \left(\Gamma(1/\beta + 1) \mathcal{H}_{\alpha_i} \frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)} \right) \Psi(u).$$

ii) If $\gamma_1 = 0, \gamma_2 \in (0, \infty]$, then

$$\mathbb{P} \left\{ \sup_{(s,t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim \mathcal{H}_{\alpha_1} \left(\frac{\gamma_2}{\theta} \right)^{1/\beta} \mathcal{I}_\beta \frac{\overleftarrow{v}_1(1/u)}{\overleftarrow{\rho}_1(1/u)} \Psi(u),$$

iii) If $\gamma_1 = 0, \gamma_2 = \infty$, then

$$\mathbb{P} \left\{ \sup_{(s,t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim 2\Gamma(1/\beta + 1) (M_\beta)^{-1/\beta} \mathcal{H}_{\alpha_1} \frac{\overleftarrow{v}_1(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})} \Psi(u),$$

iv) If $\gamma_1 \in (0, \infty], \gamma_2 = \infty$, then

$$\mathbb{P} \left\{ \sup_{(s,t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim \widehat{\mathcal{P}}_{\alpha_1}^{\gamma_1 M_\beta} \Psi(u).$$

Remark 3.6. Analogously to the Case 2, the assumption that $\eta = 0$, $\theta \in (0, \infty)$ excludes case $\gamma_1 \in (0, \infty], \gamma_2 \in [0, \infty)$.

◇ **Case 4.** Here we still assume that $A = I$ but there are no restrictions on the invertible B .

Theorem 3.7. Suppose that (14) and (15) hold with $A = I$ and B an invertible matrix, and (16) is satisfied with $\eta, \theta \in (0, \infty)$.

i) If $\gamma_1 = \gamma_2 = 0$, then

$$(21) \quad \mathbb{P} \left\{ \sup_{(s,t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim \frac{4}{|\det(B)|} \prod_{i=1}^2 \left(\Gamma(1/\beta_i + 1) \mathcal{H}_{\alpha_i} \frac{\overleftarrow{v}_i(1/u)}{\rho_i(1/u)} \right) \Psi(u).$$

ii) If $\gamma_1, \gamma_2 \in (0, \infty)$ or $\gamma_1 = \gamma_2 = \infty$, then

$$(22) \quad \mathbb{P} \left\{ \sup_{(s,t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim \widehat{\mathcal{P}}_{\alpha_1}^{\gamma_1, \gamma_1 \theta, B_{\eta, \alpha_1}} \Psi(u),$$

where $\widehat{\mathcal{P}}_{\alpha_1}^{\gamma_1, \theta \gamma_1, B_{\eta, \alpha_1}} = 1$ if $\gamma_1 = \gamma_2 = \infty$ and $B_{\eta, \alpha_1} = \begin{pmatrix} b_{11} & b_{12} \eta^{-1/\alpha_1} \\ b_{21} & b_{22} \eta^{-1/\alpha_1} \end{pmatrix}$.

3.1.1. *Discussion.* As mentioned above, all other cases for $\text{rank}(A) = 2$ can be reduced to the analysis of the field of one of types covered by Case 1-4. For the sake of transparency, let us first consider $A = I$ and B such that exactly one element b_{ij} equals to 0. With no loss of generality, by a change of variables, we can assume that

$$B = \begin{pmatrix} 1 & b_{12} \\ 0 & 1 \end{pmatrix}, \quad b_{12} \neq 0.$$

Then the following holds:

◇ $\theta = \infty$: The asymptotics of (13) in this case is covered by Case 1 above, since by Lemma 6.3 we obtain

$$v_1^2(|s + b_{12}t|) + v_2^2(|t|) \sim v_1^2(|s|) + v_2^2(|t|), \quad s, t \rightarrow 0.$$

◇ $\eta = \infty$: Let $Z(s, t) = X(s - b_{12}t, t)$, which is a *locally additive* Gaussian random field. Indeed, it follows from Lemma 6.3 that

$$1 - r_Z(s, t, s_1, t_1) \sim \rho_1^2(|s - s_1 - b_{12}(t - t_1)|) + \rho_2^2(|t - t_1|) \sim \rho_1^2(|s - s_1|) + \rho_2^2(|t - t_1|), \quad s, t, s_1, t_1 \rightarrow 0,$$

$$\text{and } 1 - \sigma_Z(s, t) \sim v_1^2(|s|) + v_2^2(|t|), \quad s, t \rightarrow 0.$$

◇ $\theta = 0, \eta = 0$: Let $Z(s, t) = X(s, \frac{t-s}{b_{12}})$. Then, again by Lemma 6.3, Z is a *locally additive* Gaussian random field with

$$1 - r_Z(s, t, s_1, t_1) \sim \rho_1^2(|s - s_1|) + |b_{12}|^{-\alpha_2} \rho_2^2(|t - t_1|), \quad s, t, s_1, t_1 \rightarrow 0,$$

$$\text{and } 1 - \sigma_Z(s, t) \sim |b_{12}|^{-\beta_2} v_2^2(|s|) + v_1^2(|t|), \quad s, t \rightarrow 0.$$

◇ $\theta = 0, \eta \in (0, \infty)$: This is covered by Case 2 above.

◇ $\theta \in (0, \infty), \eta = 0$: This is covered by Case 3 above.

◇ $\theta \in (0, \infty), \eta \in (0, \infty)$: This is covered by Case 4 above.

Next, let $A = I$ and $b_{ij} \neq 0$ for $i, j = 1, 2$. With no loss of generality we can assume that

$$B = \begin{pmatrix} 1 & b_{12} \\ b_{21} & 1 \end{pmatrix}, \quad b_{12}b_{21} \neq 0.$$

Let us observe that $\det(B) = 1 - b_{12}b_{21} \neq 0$, which will be used in several places below. Then the following holds:

◇ $\theta = 0, \eta = 0$: Let $Z(s, t) = X(s, \frac{t-s}{b_{12}})$. Again by Lemma 6.3 Z is a *locally additive* Gaussian random field

$$1 - r_Z(s, t, s_1, t_1) \sim \rho_1^2(|s - s_1|) + |b_{12}|^{-\alpha_2} \rho_2^2(|t - t_1|), \quad s, t, s_1, t_1 \rightarrow 0,$$

$$\text{and } 1 - \sigma_Z(s, t) \sim \left| \frac{\det B}{b_{12}} \right|^{\beta_2} v_2^2(|s|) + v_1^2(|t|), \quad s, t \rightarrow 0.$$

◇ $\theta = 0, \eta \in (0, \infty)$: This is Case 2 with v_2^2 replaced by $|\det(B)|^{\beta_2} v_2^2$. Indeed, by Lemma 6.3, we have

$$\begin{aligned} v_1^2(|s + b_{12}t|) + v_2^2(|b_{21}s + t|) &= v_1^2(|s + b_{12}t|) + v_2^2(|b_{21}(s + b_{12}t) + (1 - b_{12}b_{21})t|) \\ &\sim v_1^2(|s + b_{12}t|) + |\det(B)|^{\beta_2} v_2^2(|t|), \quad s, t \rightarrow 0. \end{aligned}$$

- ◇ $\theta = 0, \eta = \infty$: Let $Z(s, t) = X(s - b_{12}t, t)$. Again, by Lemma 6.3, Z is a *locally additive* Gaussian random field with

$$1 - r_Z(s, t, s_1, t_1) \sim \rho_1^2(|s - s_1|) + \rho_2^2(|t - t_1|), \quad s, t, s_1, t_1 \rightarrow 0,$$

$$\text{and } 1 - \sigma_Z(s, t) \sim v_1^2(|s|) + v_2^2(|b_{21}s + (1 - b_{12}b_{21})t|) \sim v_1^2(|s|) + |\det(B)|^{\beta_2} v_2^2(|t|), \quad s, t \rightarrow 0.$$

- ◇ $\theta \in (0, \infty), \eta = 0$: Let $Z(s, t) = X(s, t - b_{21}s)$. This is Case 3 with

$$1 - r_Z(s, t, s_1, t_1) \sim \rho_1^2(|s - s_1|) + \rho_2^2(|t - t_1|), \quad s, t, s_1, t_1 \rightarrow 0,$$

$$\text{and } 1 - \sigma_Z(s, t) \sim |\det(B)|^{\beta_1} v_1^2(|s + b_{12}(\det(B))^{-1}t|) + v_2^2(|t|), \quad s, t \rightarrow 0.$$

- ◇ $\theta \in (0, \infty), \eta \in (0, \infty)$: This is covered by Case 4.

- ◇ $\theta \in (0, \infty), \eta = \infty$: Let $Z(s, t) = X(\frac{t-s}{b_{21}}, s)$. This is Case 3 with

$$1 - r_Z(s, t, s_1, t_1) \sim \rho_2^2(|s - s_1|) + |b_{21}|^{-\alpha_1} \rho_1^2(|t - t_1|), \quad s, t, s_1, t_1 \rightarrow 0,$$

$$\text{and } 1 - \sigma_Z(s, t) \sim \left| \frac{\det(B)}{b_{21}} \right|^{\beta_1} v_1^2(|s + (-\det(B))^{-1}t|) + v_2^2(|t|), \quad s, t \rightarrow 0.$$

- ◇ $\theta = \infty, \eta = 0$: Let $Z(s, t) = X(s, t - b_{21}s)$. This is a *locally additive* Gaussian random field with v_1^2 substituted by $|\det(B)|^{\beta_1} v_1^2$.

- ◇ $\theta = \infty, \eta \in (0, \infty)$: By Lemma 6.3 we have that this is Case 2 with

$$1 - r_X(s, t, s_1, t_1) \sim \rho_1^2(|s - s_1|) + \rho_2^2(|t - t_1|), \quad s, t, s_1, t_1 \rightarrow 0,$$

and

$$\begin{aligned} 1 - \sigma_X(s, t) &= v_2^2(|b_{21}s + t|) + v_1^2(|b_{21}^{-1}(b_{21}s + t) + (b_{12} - b_{21}^{-1})t|) \\ &\sim v_2^2(|b_{21}s + t|) + v_1^2((b_{12} - b_{21}^{-1})t) \\ &\sim |b_{21}|^{\beta_2} v_2^2(|s + (b_{21})^{-1}t|) + \left| \frac{\det(B)}{b_{21}} \right|^{\beta_1} v_1^2(|t|), \quad s, t \rightarrow 0. \end{aligned}$$

- ◇ $\theta = \infty, \eta = \infty$: Let $Z(s, t) = X(\frac{s-t}{b_{21}}, t)$. We have that Z is a *locally additive* Gaussian random field with

$$1 - r_Z(s, t, s_1, t_1) \sim |b_{21}|^{-\alpha_1} \rho_1^2(|s - s_1|) + \rho_2^2(|t - t_1|), \quad s, t, s_1, t_1 \rightarrow 0,$$

$$\text{and } 1 - \sigma_Z(s, t) \sim v_2^2(|s|) + \left| \frac{\det(B)}{b_{21}} \right|^{\beta_1} v_1^2(|t|), \quad s, t \rightarrow 0.$$

3.2. Scenario II: $\text{rank}(A) = 1$. Suppose that $\text{rank}(A) = 1$. Clearly it suffices to consider Gaussian random fields with covariance function that satisfies (14) with $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and variance function satisfying (15). We begin with the analysis of two special cases, to which all other structures of field X can be reduced.

- ◇ Case 5. Here we shall assume that (14) and (15) are satisfied with

$$(23) \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = I.$$

Theorem 3.8. Suppose that (23) holds.

- i) If $\gamma_1 = 0$, then

$$\mathbb{P} \left\{ \sup_{(s,t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim 2\Gamma(1/\beta_1 + 1) \mathcal{H}_{\alpha_1} \frac{\overleftarrow{v}_1(1/u)}{\overleftarrow{\rho}_1(1/u)} \Psi(u).$$

- ii) If $\gamma_1 \in (0, \infty]$, then

$$(24) \quad \mathbb{P} \left\{ \sup_{(s,t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim \widehat{\mathcal{P}}_{\alpha_1}^{\gamma_1} \Psi(u).$$

- ◇ Case 6. Here we shall assume that (14) and (15) are satisfied with

$$(25) \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & b_{12} \\ 0 & 1 \end{pmatrix}, \text{ and } b_{12} \neq 0.$$

Theorem 3.9. *Suppose that (25) holds and (16) is satisfied with $\theta \in (0, \infty)$.*

i) *If $\gamma_1 = 0$, then*

$$(26) \quad \mathbb{P} \left\{ \sup_{(s,t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim 2(M_{\beta_1})^{-1/\beta_1} \Gamma(1/\beta_1 + 1) \mathcal{H}_{\alpha_1} \frac{\overleftarrow{v}_1(1/u)}{\overleftarrow{\rho}_1(1/u)} \Psi(u).$$

ii) *If $\gamma_1 \in (0, \infty]$, then*

$$(27) \quad \mathbb{P} \left\{ \sup_{(s,t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \sim \widehat{\mathcal{P}}_{\alpha_1}^{\gamma_1 M_{\beta_1}} \Psi(u)$$

with M_β defined in (20).

3.2.1. *Discussion.* Having analyzed the above special cases, we are now in position to give the asymptotics of (13) for general structure of X . Suppose first, analogously to Scenario I, that X satisfies (14) and (15) with $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and exactly one element of matrix B equals 0. With no loss of generality we can assume that $B = \begin{pmatrix} 1 & b_{12} \\ 0 & 1 \end{pmatrix}$, $b_{12} \neq 0$. Then the following holds.

◇ $\theta = 0$: Let $Z(s, t) = X(s, \frac{t-s}{b_{12}})$. Then, by Lemma 6.3, this is Case 5 with

$$1 - r_Z(s, t, s_1, t_1) \sim \rho_1^2(|s - s_1|), s, t, s_1, t_1 \rightarrow 0, \quad 1 - \sigma_Z(s, t) \sim |b_{12}|^{-\beta_2} v_2^2(|s|) + v_1^2(|t|), \quad s, t \rightarrow 0.$$

◇ $\theta \in (0, \infty)$: This is case 6.

◇ $\theta = \infty$: The asymptotics of (13) in this case is the same as the asymptotis derived in Case 5. Indeed, by Lemma 6.3, we have

$$v_1^2(|s + b_{12}t|) + v_2^2(|t|) \sim v_1^2(|s|) + v_2^2(|t|), \quad s, t \rightarrow 0.$$

Finally we discuss the other case where the matrix B is such that $b_{ij} \neq 0$ for $i, j = 1, 2$. Again with no loss of generality we can assume that $B = \begin{pmatrix} 1 & b_{12} \\ b_{21} & 1 \end{pmatrix}$, $b_{12}, b_{21} \neq 0$. Then the following holds with $\det(B) = 1 - b_{12}b_{21} \neq 0$:

◇ $\theta = 0$: Let $Z(s, t) = X(s, \frac{t-s}{b_{12}})$. This is covered by Case 5.

$$1 - r_Z(s, t, s_1, t_1) \sim \rho_1^2(|s - s_1|), s, t, s_1, t_1 \rightarrow 0, \quad 1 - \sigma_Z(s, t) \sim \left| \frac{\det(B)}{b_{12}} \right|^{\beta_2} v_2^2(|s|) + v_1^2(|t|), \quad s, t \rightarrow 0.$$

◇ $\theta \in (0, \infty)$: Let $Z(s, t) = X(s, t - b_{21}s)$. Then, by Lemma 6.3, Z is as in Case 6 with

$$1 - r_Z(s, t, s_1, t_1) \sim \rho_1^2(|s - s_1|), \quad s, t, s_1, t_1 \rightarrow 0, \quad 1 - \sigma_Z(s, t) \sim |\det(B)|^{\beta_1} v_1^2(|s + b_{12}(\det(B))^{-1}t|) + v_2^2(|t|), \quad s, t \rightarrow 0.$$

◇ $\theta = \infty$: Let $Z(s, t) = X(s, t - b_{21}s)$. This is Case 5 with

$$1 - r_Z(s, t, s_1, t_1) \sim \rho_1^2(|s - s_1|), s, t, s_1, t_1 \rightarrow 0, \quad 1 - \sigma_Z(s, t) \sim |\det(B)|^{\beta_1} v_1^2(|s|) + v_2^2(|t|), \quad s, t \rightarrow 0.$$

4. PROOFS

In the rest of this section by $\mathbb{Q}, \mathbb{Q}_i > 0, i = 1, 2, \dots$ we denote constants that may differ from line to line.

Proof of Theorem 2.1 We set, for $u > 1$ and $\xi(u) := u^{-1} \ln u$

$$E_u = [0, \overleftarrow{v}(\xi(u))], \quad I_k(u) = [kS\overleftarrow{\rho}(u^{-1}), (k+1)S\overleftarrow{\rho}(u^{-1})], k \in \mathbb{N} \cup \{0\}$$

and, for given $\varepsilon \in (0, 1/2)$, define

$$u_{k,\varepsilon}^- = u(1 + (1 - \varepsilon) \inf_{t \in I_k(u)} v^2(t)), \quad u_{k,\varepsilon}^+ = u(1 + (1 + \varepsilon) \sup_{t \in I_k(u)} v^2(t)), \quad N(u) = \left\lceil \frac{\overleftarrow{v}(\xi(u))}{\overleftarrow{\rho}(u^{-1})S} \right\rceil + 1.$$

For $L > 0$ sufficiently small

$$(28) \quad \mathbb{E} \{ \overline{X}(t) - \overline{X}(t)^2 \} \leq 2(1 - r(s, t)) \leq 4\rho^2(|t - s|) \leq \mathbb{Q}|t - s|^{\alpha/2}, \quad s, t \in [0, L],$$

which ensures the Hölder condition in a neighborhood of 0. By the fact that $\sup_{t \in [\bar{v}(\xi(u)), T]} \sigma(t) \leq 1 - \mathbb{Q}(\xi(u))^2$ for u sufficiently large, (28), Theorem 8.1 in [28], we have

$$\mathbb{P} \left\{ \sup_{t \in [\bar{v}(\xi(u)), L]} X(t) > u \right\} \leq \mathbb{Q} T u^{4/\alpha} \Psi \left(\frac{u}{1 - \mathbb{Q}(\xi(u))^2} \right).$$

Moreover, in light of Borell inequality (see e.g., [35]) and the fact that $\sup_{t \in [L, T]} \sigma(t) \leq 1 - \delta$ with $\delta > 0$,

$$\mathbb{P} \left\{ \sup_{t \in [\bar{v}(\xi(u)), L]} X(t) > u \right\} \leq e^{-\frac{(u-a)^2}{2(1-\delta)}}$$

with $a = \mathbb{E} \left(\sup_{t \in [0, T]} X(t) \right)$.

Consequently, for all large u we have

$$(29) \quad \pi(u) \leq \mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \leq \pi(u) + \mathbb{Q} T u^{4/\alpha} \Psi \left(\frac{u}{1 - \mathbb{Q}(\xi(u))^2} \right),$$

where $\pi(u) = \mathbb{P} \left\{ \sup_{t \in [0, \bar{v}(\xi(u))]} X(t) > u \right\}$.

Next we give the exact asymptotics of $\pi(u)$ subject to three different scenarios.

Case i) $\gamma = 0$. For any u positive we have

$$(30) \quad \sum_{k=0}^{N(u)-1} \mathbb{P} \left\{ \sup_{t \in I_k(u)} X(t) > u \right\} - \sum_{i=1}^2 \Lambda_i(u) \leq \pi(u) \leq \sum_{k=0}^{N(u)} \mathbb{P} \left\{ \sup_{t \in I_k(u)} X(t) > u \right\},$$

where

$$\Lambda_1(u) = \sum_{k=0}^{N(u)} \mathbb{P} \left\{ \sup_{t \in I_k(u)} X(t) > u, \sup_{t \in I_{k+1}(u)} X(t) > u \right\},$$

and

$$\Lambda_2(u) = \sum_{0 \leq k, l \leq N(u), l \geq k+2} \mathbb{P} \left\{ \sup_{t \in I_k(u)} X(t) > u, \sup_{t \in I_l(u)} X(t) > u \right\}.$$

The main difference in comparison with the proofs of the classical cases considered in the literature, as e.g., in [28], is contained in the approximation given below. By uniform convergence theorem (UCT) for regularly varying functions, see e.g., [3], we have

$$\sup_{s, t \in I_k(u), 1 \leq k \leq N(u)} \left| \frac{v^2(s)}{v^2(t)} - \left(\frac{s}{t} \right)^\alpha \right| \rightarrow 0, \quad u \rightarrow \infty,$$

which implies that for any $\epsilon > 0$ and for u sufficiently large,

$$\frac{v^2(s)}{v^2(t)} \geq \left(\frac{k}{k+1} \right)^\alpha - \epsilon/2, \quad s, t \in I_k(u), 1 \leq k \leq N(u).$$

Thus for any $\epsilon > 0$, there exists $k_\epsilon \in \mathbb{N}$ such that

$$\inf_{t \in I_k(u)} v^2(t) \geq (1 - \epsilon) \sup_{t \in I_k(u)} v^2(t), \quad k_\epsilon \leq k \leq N(u).$$

Let $X_{u,k}(t) = \overline{X}(kS\bar{\rho}(u^{-1}) + t), t \in I_0(u)$ with $k \in \mathcal{K}_u = \{k, 0 \leq k \leq N(u)\}$ and $h_k(u) = u_{k,\epsilon}^-$. In light of Lemma 5.1, we have

$$(31) \quad \lim_{u \rightarrow \infty} \sup_{0 \leq k \leq N(u)} \left| (\Psi(u_{k,\epsilon}^-))^{-1} \mathbb{P} \left\{ \sup_{t \in I_0(u)} X_{u,k}(t) > u_{k,\epsilon}^- \right\} - \mathcal{H}_\alpha[0, S] \right| = 0.$$

Consequently, as $u \rightarrow \infty$

$$\begin{aligned} \sum_{k=0}^{N(u)} \mathbb{P} \left\{ \sup_{t \in I_k(u)} X(t) > u \right\} &\leq \sum_{k=0}^{N(u)} \mathbb{P} \left\{ \sup_{t \in I_k(u)} \overline{X}(t) > u_{k,\epsilon}^- \right\} \\ &\leq \sum_{k=0}^{N(u)} \mathbb{P} \left\{ \sup_{t \in I_0(u)} X_{u,k}(t) > u_{k,\epsilon}^- \right\} \end{aligned}$$

$$\begin{aligned}
& \sim \sum_{k=0}^{N(u)} \mathcal{H}_\alpha[0, S] \Psi(u_{k, \epsilon}^-) \\
& \sim \mathcal{H}_\alpha[0, S] \Psi(u) \sum_{k=0}^{N(u)} e^{-u^2(1-\epsilon) \inf_{t \in I_k(u)} v^2(t)}.
\end{aligned}$$

Further by Lemma 6.2

$$\begin{aligned}
\sum_{k=0}^{N(u)} \mathbb{P} \left\{ \sup_{t \in I_k(u)} X(t) > u \right\} & \leq \mathcal{H}_\alpha[0, S] \Psi(u) \left(k_\epsilon + \frac{1}{\overleftarrow{\rho}(u^{-1})S} \sum_{k=k_\epsilon}^{N(u)} \int_{t \in I_k(u)} e^{-(1-\epsilon)^2 u^2 v^2(t)} dt \right), \\
& \sim \mathcal{H}_\alpha[0, S] \left(k_\epsilon + \frac{1}{\overleftarrow{\rho}(u^{-1})S} \int_0^{\overleftarrow{v}(\xi(u))} e^{-(1-\epsilon)^2 u^2 v^2(t)} dt \right) \Psi(u) \\
& \sim \Gamma(1/\beta + 1) \mathcal{H}_\alpha \frac{\overleftarrow{v}(u^{-1})}{\overleftarrow{\rho}(u^{-1})} \Psi(u), \quad u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0.
\end{aligned}$$

Similarly, we obtain

$$(32) \quad \sum_{k=0}^{N(u)-1} \mathbb{P} \left\{ \sup_{t \in I_k(u)} X(t) > u \right\} \geq \Gamma(1/\beta + 1) \mathcal{H}_\alpha \frac{\overleftarrow{v}(u^{-1})}{\overleftarrow{\rho}(u^{-1})} \Psi(u) (1 + o(1)), \quad u \rightarrow \infty, S \rightarrow \infty.$$

Next we focus on $\Lambda_i(u)$, $i = 1, 2$. Let $\hat{u}_{k, -\epsilon} = \min(u_{k, -\epsilon}^-, u_{k+1, -\epsilon}^-)$. Then by (31) the fact that

$$\begin{aligned}
& \mathbb{P} \left\{ \sup_{t \in I_k(u)} X(t) > u, \sup_{t \in I_{k+1}(u)} X(t) > u \right\} \\
& \leq \mathbb{P} \left\{ \sup_{t \in I_k(u)} \overline{X}(t) > \hat{u}_{k, -\epsilon} \right\} + \mathbb{P} \left\{ \sup_{t \in I_{k+1}(u)} \overline{X}(t) > \hat{u}_{k, -\epsilon} \right\} - \mathbb{P} \left\{ \sup_{t \in I_k(u) \cup I_{k+1}(u)} \overline{X}(t) > \hat{u}_{k, -\epsilon} \right\},
\end{aligned}$$

we have

$$\lim_{S \rightarrow \infty} \lim_{u \rightarrow \infty} \sup_{0 \leq k \leq N(u)} \frac{\mathbb{P} \left\{ \sup_{t \in I_k(u)} X(t) > u, \sup_{t \in I_{k+1}(u)} X(t) > u \right\}}{\mathcal{H}_\alpha[0, S] \Psi(\hat{u}_{k, -\epsilon})} \leq \lim_{S \rightarrow \infty} \left(2 - \frac{\mathcal{H}_\alpha[0, 2S]}{\mathcal{H}_\alpha[0, S]} \right) = 0.$$

Therefore,

$$\begin{aligned}
\Lambda_1(u) &= o(1) \sum_{k=0}^{N(u)} \mathcal{H}_\alpha[0, S] \Psi(\hat{u}_{k, -\epsilon}) \leq o(1) \sum_{k=0}^{N(u)} 2 \mathcal{H}_\alpha[0, S] \Psi(u_{k, -\epsilon}) \\
&= o \left(\frac{\overleftarrow{v}(u^{-1})}{\overleftarrow{\rho}(u^{-1})} \Psi(u) \right), \quad u \rightarrow \infty, S \rightarrow \infty.
\end{aligned}$$

By (8) and applying Lemma 5.4 in Appendix, we have (note that below k, l take values up to N_u , therefore an uniform upper bound for approximating the summands derived in Lemma 5.4 is essential)

$$\begin{aligned}
\Lambda_2(u) &\leq \sum_{0 \leq k, l \leq N(u), l \geq k+2} \mathbb{P} \left\{ \sup_{t \in I_k(u)} \overline{X}(t) > u_{k, -\epsilon}^-, \sup_{t \in I_l(u)} \overline{X}(t) > u_{l, -\epsilon}^- \right\} \\
&\leq \sum_{0 \leq k, l \leq N(u), l \geq k+2} \mathbb{Q} S^2 \Psi(\hat{u}_{k, l, -\epsilon}) e^{-\mathbb{Q}_1 |(l-k)S|^{\alpha/2}} \\
&\leq \mathbb{Q} S^2 \sum_{0 \leq k \leq N(u)} \Psi(u_{k, -\epsilon}^-) \sum_{l=1}^{\infty} e^{-\mathbb{Q}_1 (lS)^{\alpha/2}} \\
&\leq \mathbb{Q} S^2 e^{-\mathbb{Q}_2 S^{\alpha/2}} \sum_{k=0}^{N(u)} \Psi(u_{k, -\epsilon}^-) \\
&= o \left(\frac{\overleftarrow{v}(u^{-1})}{\overleftarrow{\rho}(u^{-1})} \Psi(u) \right), \quad u \rightarrow \infty, S \rightarrow \infty,
\end{aligned}$$

with $\hat{u}_{k, l, -\epsilon} = \min(u_{k, -\epsilon}^-, u_{l, -\epsilon}^-)$. By the above calculations both $\Lambda_1(u)$ and $\Lambda_2(u)$ are negligible. Hence the results displayed in (29)-(33) establish the claim.

Case ii) $\gamma \in (0, \infty]$. The proof of this case is the same as the proof of the corresponding counterpart of Theorem D2 in [28], with the exception that

$$\pi(u) \sim \mathbb{P} \left\{ \sup_{t \in I_0(u)} X(t) > u \right\} \sim \mathcal{P}_\alpha^\gamma[0, S] \Psi(u), \quad u \rightarrow \infty,$$

where the last asymptotics follows by Lemma 5.1. This completes the proof. \square

4.1. **Proofs of Theorems 3.1, 3.3, 3.5, 3.7, 3.8 and 3.9.** Define next for S, u positive

$$\begin{aligned} I_{k,l}(u) &= [\check{\rho}_1(u^{-1})kS, \check{\rho}_1(u^{-1})(k+1)S] \times [\check{\rho}_2(u^{-1})lS, \check{\rho}_2(u^{-1})(l+1)S], k, l \in \mathbb{N} \cup \{0\}, \\ I_k(u) &= [\check{\rho}_1(u^{-1})kS, \check{\rho}_1(u^{-1})(k+1)S], \quad J_k(u) = [\check{\rho}_2(u^{-1})kS, \check{\rho}_2(u^{-1})(k+1)S], \end{aligned}$$

for $k \in \mathbb{N} \cup \{0\}$, and

$$N_1(u) = \left\lfloor \frac{\check{\rho}_1(u^{-1} \ln u)}{\check{\rho}_1(u^{-1})S} \right\rfloor, \quad N_2(u) = \left\lfloor \frac{\check{\rho}_2(u^{-1} \ln u)}{\check{\rho}_2(u^{-1})S} \right\rfloor.$$

Additionally, let

$$\mathbb{V}_1(u) = \{(k, l, k_1, l_1) : -N_1(u) - 2 \leq k \leq k_1 \leq N_1(u) + 1, -N_2(u) - 2 \leq l, l_1 \leq N_2(u) + 1, I_{k,l} \cap I_{k_1,l_1} = \emptyset\},$$

$$\mathbb{V}_2(u) = \{(k, l, k_1, l_1) : -N_1(u) - 2 \leq k \leq k_1 \leq N_1(u) + 1, -N_2(u) - 2 \leq l, l_1 \leq N_2(u) + 1, (k, l) \neq (k_1, l_1), I_{k,l} \cap I_{k_1,l_1} \neq \emptyset\},$$

$$u_{k,l,\epsilon}^- = u(1 + (1 - \epsilon) \inf_{(s,t) \in I_{k,l}(u)} (v_1^2(|b_{11}s + b_{12}t|) + v_2^2(|b_{21}s + b_{22}t|))),$$

$$u_{k,l,\epsilon}^+ = u(1 + (1 + \epsilon) \sup_{(s,t) \in I_{k,l}(u)} (v_1^2(|b_{11}s + b_{12}t|) + v_2^2(|b_{21}s + b_{22}t|))),$$

$$u_{k,\epsilon}^{1,-} = u(1 + (1 - \epsilon) \inf_{s \in I_k(u)} v_1^2(|s|)), \quad u_{k,\epsilon}^{1,+} = u(1 + (1 + \epsilon) \sup_{s \in I_k(u)} v_1^2(|s|)),$$

$$u_{l,\epsilon}^{2,-} = u(1 + (1 - \epsilon) \inf_{s \in J_l(u)} v_2^2(|s|)), \quad u_{l,\epsilon}^{2,+} = u(1 + (1 + \epsilon) \sup_{s \in J_l(u)} v_2^2(|s|)), k, l \in \mathbb{Z},$$

where $u_{k,l,\epsilon}^\pm$ varies according to B .

In what follows for a given Gaussian random field Z we write \overline{Z} for the standardised random field.

The general strategy of proofs of Theorems 3.1, 3.3, 3.5, 3.7, 3.8 and 3.9 agrees from the double-sum technique developed for Gaussian random fields in e.g., [28]. However the variance-covariance structure of some cases substantially differs from the families of Gaussian random fields analyzed in [28] and requires a case-specific approach, on which we focus below.

Observe that for all Cases 1-6

$$(33) \quad \pi_1(u) \leq \mathbb{P} \left\{ \sup_{(s,t) \in [-T_1, T_1] \times [-T_2, T_2]} X(s, t) > u \right\} \leq \pi_1(u) + \mathbb{P} \left\{ \sup_{(s,t) \in ([-T_1, T_1] \times [-T_2, T_2]) \setminus D_u} X(s, t) > u \right\},$$

where

$$\pi_1(u) = \mathbb{P} \left\{ \sup_{(s,t) \in D_u} X(s, t) > u \right\}, \text{ with } D_u = [-\check{\rho}_1(u^{-1} \ln u), \check{\rho}_1(u^{-1} \ln u)] \times [-\check{\rho}_2(u^{-1} \ln u), \check{\rho}_2(u^{-1} \ln u)].$$

For Case 1-Case 3 and Case 5-Case 6, by (15) for u sufficiently large we have

$$\sup_{(s,t) \in ([-T_1, T_1] \times [-T_2, T_2]) \setminus D_u} \sigma(s, t) \leq 1 - \mathbb{Q}u^{-2} \ln^2 u.$$

For Case 4, in light of (15) and Lemma 6.4, we have

$$\sup_{(s,t) \in ([-T_1, T_1] \times [-T_2, T_2]) \setminus D_u} \sigma(s, t) \leq 1 - \mathbb{Q} \inf_{(s,t) \in ([-T_1, T_1] \times [-T_2, T_2]) \setminus D_u} (v_1^2(|s|) + v_2^2(|t|)) \leq 1 - \mathbb{Q}u^{-2} \ln^2 u.$$

It follows by the fact that $(0, 0)$ is the unique maximizer of σ , Theorem 8.1 in [28] and Borell theorem that

$$(34) \quad \mathbb{P} \left\{ \sup_{(s,t) \in ([-T_1, T_1] \times [-T_2, T_2]) \setminus D_u} X(s, t) > u \right\} \leq \mathbb{Q}T_1T_2u^{4/\alpha_1+4/\alpha_2} \Psi \left(\frac{u}{1 - 2u^{-2} \ln^2 u} \right).$$

Therefore, for all Cases 1-6 we focus on the asymptotics of $\pi_1(u)$ as $u \rightarrow \infty$, proving that it delivers the asymptotics of (13) as $u \rightarrow \infty$.

Proof of Theorem 3.1

Case i). Suppose that $\gamma_1 = \gamma_2 = 0$. For any $0 < \epsilon < 1/2$ and u large enough we have

$$(35) \quad \pi_{1,\epsilon}(u) - \sum_{i=1}^2 \Lambda'_i(u) \leq \pi_1(u) \leq \pi_{1,-\epsilon}(u),$$

with

$$\begin{aligned} \pi_{1,\pm\epsilon}(u) : &= \sum_{k=-N_1(u)\pm 2}^{N_1(u)\mp 1} \sum_{l=-N_2(u)\pm 2}^{N_2(u)\mp 1} \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^{\pm} \right), \\ \Lambda'_1(u) : &= \sum_{(k,l,k_1,l_1) \in \mathbb{V}_1(u)} \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^-, \sup_{(s,t) \in I_{k_1,l_1}(u)} \overline{X}(s,t) > u_{k_1,l_1,\epsilon}^- \right), \\ \Lambda'_2(u) : &= \sum_{(k,l,k_1,l_1) \in \mathbb{V}_2(u)} \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^-, \sup_{(s,t) \in I_{k_1,l_1}(u)} \overline{X}(s,t) > u_{k_1,l_1,\epsilon}^- \right). \end{aligned}$$

By UCT, for any $\epsilon > 0$, there exist two constants $k_\epsilon, l_\epsilon \in \mathbb{N}$ such that

$$(36) \quad \inf_{t \in I_k(u)} v_1^2(t) \geq (1-\epsilon) \sup_{t \in I_k(u)} v_1^2(t), \quad \inf_{t \in J_l(u)} v_2^2(t) \geq (1-\epsilon) \sup_{t \in J_l(u)} v_2^2(t),$$

hold for $k_\epsilon \leq |k| \leq N_1(u) + 2, l_\epsilon \leq |l| \leq N_2(u) + 2$. Let

$$X_{u,k,l}(s,t) = \overline{X}(kS^{\leftarrow} \overline{\rho}_1(u^{-1}) + s, lS^{\leftarrow} \overline{\rho}_2(u^{-1}) + t), \mathcal{K}_u = \{(k,l), |k| \leq N_1(u) + 2, |l| \leq N_2(u) + 2\},$$

$$h_{k,l}(u) = u_{k,l,\epsilon}^-, \mathcal{E}_u = I_{0,0}(u), d_u = 0.$$

One can easily check that conditions of Lemma 5.2 are satisfied implying that

$$(37) \quad \lim_{u \rightarrow \infty} \sup_{(k,l) \in \mathcal{K}_u} \left| (\Psi(u_{k,l,\epsilon}^-))^{-1} \mathbb{P} \left(\sup_{(s,t) \in I_{0,0}(u)} \overline{X}(kS^{\leftarrow} \overline{\rho}_1(u^{-1}) + s, lS^{\leftarrow} \overline{\rho}_2(u^{-1}) + t) > u_{k,l,\epsilon}^- \right) - \prod_{i=1}^2 \mathcal{H}_{\alpha_i}[0, S] \right| = 0.$$

Further, using Lemma 6.2 we have

$$\begin{aligned} \pi_{1,-\epsilon}(u) &= \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \mathbb{P} \left(\sup_{(s,t) \in I_{0,0}(u)} \overline{X}(kS^{\leftarrow} \overline{\rho}_1(u^{-1}) + s, lS^{\leftarrow} \overline{\rho}_2(u^{-1}) + t) > u_{k,l,\epsilon}^- \right) \\ &\sim \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \prod_{i=1}^2 \mathcal{H}_{\alpha_i}[0, S] \Psi(u_{k,l,\epsilon}^-) \\ &\sim \prod_{i=1}^2 \mathcal{H}_{\alpha_i}[0, S] \Psi(u) \left(R_1(u) + R_2(u) + \sum_{k_\epsilon \leq |k| \leq N_1(u)+2} \frac{1}{\overline{\rho}_1(u^{-1})S} \int_{s \in I_k(u)} e^{-(1-\epsilon)^2 u^2 v_1^2(|t|)} dt \right. \\ &\quad \times \left. \sum_{l_\epsilon \leq |l| \leq N_2(u)+2} \frac{1}{\overline{\rho}_2(u^{-1})S} \int_{t \in J_l(u)} e^{-(1-\epsilon)^2 u^2 v_2^2(t)} dt \right) \\ &\sim 4 \prod_{i=1}^2 \mathcal{H}_{\alpha_i}[0, S] \Psi(u) \frac{1}{\overline{\rho}_1(u^{-1})S} \int_0^{\overline{v}_1(u^{-1} \ln u)} e^{-(1-\epsilon)^2 u^2 v_1^2(t)} dt \\ &\quad \times \frac{1}{\overline{\rho}_2(u^{-1})S} \int_0^{\overline{v}_2(u^{-1} \ln u)} e^{-(1-\epsilon)^2 u^2 v_2^2(t)} dt \\ &\sim 4(1-\epsilon)^{-1/\beta_1-1/\beta_2} \Gamma(1/\beta_1+1) \Gamma(1/\beta_2+1) S^{-2} \prod_{i=1}^2 \mathcal{H}_{\alpha_i}[0, S] \frac{\overline{v}_1(1/u)}{\overline{\rho}_1(1/u)} \frac{\overline{v}_2(1/u)}{\overline{\rho}_2(1/u)} \Psi(u) \\ (38) \quad &\sim 4\Gamma(1/\beta_1+1) \Gamma(1/\beta_2+1) \prod_{i=1}^2 \mathcal{H}_{\alpha_i} \frac{\overline{v}_1(1/u)}{\overline{\rho}_1(1/u)} \frac{\overline{v}_2(1/u)}{\overline{\rho}_2(1/u)} \Psi(u), \quad u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0, \end{aligned}$$

where

$$\begin{aligned} R_1(u) &= \sum_{|k| \leq k_\epsilon} \sum_{|l| \leq N_2(u)+2} e^{-(1-\epsilon)u^2 \inf_{s \in I_k(u)} v_1^2(|s|) - (1-\epsilon)u^2 \inf_{t \in J_l(u)} v_2^2(|t|)}, \\ R_2(u) &= \sum_{|k| \leq N_1(u)+2} \sum_{|l| \leq l_\epsilon} e^{-(1-\epsilon)u^2 \inf_{s \in I_k(u)} v_1^2(|s|) - (1-\epsilon)u^2 \inf_{t \in J_l(u)} v_2^2(|t|)}. \end{aligned}$$

Note that (38) holds since in light of Lemma 6.2 we have

$$\begin{aligned} R_1(u) &\leq (2k_\epsilon + 1) \left(2l_\epsilon + 1 + \frac{1}{\overline{\rho}_2(u^{-1})S} \int_0^{\overline{v}_2(u^{-1} \ln u)} e^{-(1-\epsilon)^2 u^2 v_2^2(t)} dt \right) \\ &\sim (2k_\epsilon + 1)(1-\epsilon)^{-1/\beta_2} \frac{\overline{v}_2(1/u)}{\overline{\rho}_2(1/u)S} = o\left(\frac{\overline{v}_1(1/u)}{\overline{\rho}_1(1/u)} \frac{\overline{v}_2(1/u)}{\overline{\rho}_2(1/u)}\right), \quad u \rightarrow \infty, \end{aligned}$$

and

$$\begin{aligned} R_2(u) &\leq (2l_\epsilon + 1) \left(2k_\epsilon + 1 + \frac{1}{\overline{\rho}_1(u^{-1})S} \int_0^{\overline{v}_1(u^{-1} \ln u)} e^{-(1-\epsilon)^2 u^2 v_1^2(t)} dt \right) \\ &\sim (2l_\epsilon + 1)(1-\epsilon)^{-1/\beta_1} \frac{\overline{v}_1(1/u)}{\overline{\rho}_1(1/u)S} \\ &= o\left(\frac{\overline{v}_1(1/u)}{\overline{\rho}_1(1/u)} \frac{\overline{v}_2(1/u)}{\overline{\rho}_2(1/u)}\right), \quad u \rightarrow \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} \pi_{1,\epsilon}(u) &\geq \sum_{k=-N_1(u)+1}^{N_1(u)-1} \sum_{l=-N_2(u)+1}^{N_2(u)-1} \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^+ \right) \\ (39) \quad &\sim 4\Gamma(1/\beta_1 + 1)\Gamma(1/\beta_2 + 1) \prod_{i=1}^2 \mathcal{H}_{\alpha_i} \frac{\overline{v}_1(1/u)}{\overline{\rho}_1(1/u)} \frac{\overline{v}_2(1/u)}{\overline{\rho}_2(1/u)} \Psi(u), \end{aligned}$$

as $u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0$.

Next we prove that both $\Lambda'_1(u), \Lambda'_2(u)$ are asymptotically negligible. From (14), applying Lemma 5.4 in the Appendix, with

$$\hat{u}_{k,l,k_1,l_1,\epsilon} = \min(u_{k,l,\epsilon}^-, u_{k_1,l_1,\epsilon}^-), \quad \beta^* = \min(\alpha_1, \alpha_2),$$

we obtain

$$\begin{aligned} \Lambda'_1(u) &\leq \mathbb{Q}S^4 \sum_{(k,l,k_1,l_1) \in \mathbb{V}_1(u)} \Psi(\hat{u}_{k,l,k_1,l_1,\epsilon}) e^{-\mathbb{Q}_1(|k-k_1|^2 + |l-l_1|^2)^{\beta^*} S^{\beta^*/2}} \\ &\leq \mathbb{Q}S^4 \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \Psi(u_{k,l,\epsilon}^-) \sum_{m+n \geq 1, m,n \geq 0} e^{-\mathbb{Q}_1(|m|^2 + |n|^2)^{\beta^*} S^{\beta^*/2}} \\ &\leq \mathbb{Q}S^4 \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \Psi(u_{k,l,\epsilon}^-) e^{-\mathbb{Q}_2 S^{\beta^*/2}} \\ (40) \quad &= o(\pi_{1,\epsilon}(u)), \quad u \rightarrow \infty, S \rightarrow \infty. \end{aligned}$$

Now we focus on $\Lambda_2(u)$. Without loss of generality, we assume that $k_1 = k + 1$. Then let, for $k_1, l_1 \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} I_{k_1,l_1}^{(1)} &= [\overline{\rho}_1(u^{-1})k_1 S, \overline{\rho}_1(u^{-1})(k_1 S + \sqrt{S})] \times [\overline{\rho}_2(u^{-1})l_1 S, \overline{\rho}_2(u^{-1})(l_1 + 1)S], \\ I_{k_1,l_1}^{(2)} &= [\overline{\rho}_1(u^{-1})(k_1 S + \sqrt{S}), \overline{\rho}_1(u^{-1})(k_1 + 1)S] \times [\overline{\rho}_2(u^{-1})l_1 S, \overline{\rho}_2(u^{-1})(l_1 + 1)S]. \end{aligned}$$

For $(k, l, k_1, l_1) \in \mathbb{V}_2(u)$, $k_1 = k + 1$, we have

$$\begin{aligned} &\mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^-, \sup_{(s,t) \in I_{k_1,l_1}(u)} \overline{X}(s,t) > u_{k_1,l_1,\epsilon}^- \right) \\ &\leq \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^-, \sup_{(s,t) \in I_{k_1,l_1}^{(1)}(u)} \overline{X}(s,t) > u_{k_1,l_1,\epsilon}^- \right) \end{aligned}$$

$$\begin{aligned}
& + \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^-, \sup_{(s,t) \in I_{k_1,l_1}^{(2)}(u)} \overline{X}(s,t) > u_{k_1,l_1,\epsilon}^- \right) \\
& := p_{k,l,k_1,l_1}^{(1)}(u) + p_{k,l,k_1,l_1}^{(2)}(u).
\end{aligned}$$

It follows from Lemma 5.2 that

$$p_{k,l,k_1,l_1}^{(1)}(u) \leq \mathbb{P} \left(\sup_{(s,t) \in I_{k_1,l_1}^{(1)}(u)} \overline{X}(s,t) > u_{k_1,l_1,\epsilon}^- \right) \sim \mathcal{H}_{\alpha_1}[0, \sqrt{S}] \mathcal{H}_{\alpha_2}[0, S] \Psi(u_{k,l,\epsilon}^-).$$

Further, since each $I_{k,l}(u) \times I_{k_1,l_1}(u)$ has at most 8 neighbors, we have that

$$\begin{aligned}
\sum_{(k,l,k_1,l_1) \in \mathbb{V}_2(u)} p_{k,l,k_1,l_1}^{(1)}(u) & \leq 8 \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \Psi(u_{k,l,\epsilon}^-) \\
& \times \left(\mathcal{H}_{\alpha_1}[0, \sqrt{S}] \mathcal{H}_{\alpha_2}[0, S] + \mathcal{H}_{\alpha_1}[0, S] \mathcal{H}_{\alpha_2}[0, \sqrt{S}] \right) \\
& \leq 8 \left(\frac{\mathcal{H}_{\alpha_1}[0, \sqrt{S}]}{\mathcal{H}_{\alpha_1}[0, S]} + \frac{\mathcal{H}_{\alpha_2}[0, \sqrt{S}]}{\mathcal{H}_{\alpha_2}[0, S]} \right) \\
& \times \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \mathcal{H}_{\alpha_1}[0, S] \mathcal{H}_{\alpha_2}[0, S] \Psi(u_{k,l,\epsilon}^-) \\
& = o(\pi_{1,\epsilon}(u)), \quad u \rightarrow \infty, S \rightarrow \infty.
\end{aligned}$$

In light of Lemma 5.4, we have

$$\begin{aligned}
\sum_{(k,l,k_1,l_1) \in \mathbb{V}_2(u)} p_{k,l,k_1,l_1}^{(2)}(u) & \leq \mathbb{Q} S^4 e^{-\mathbb{Q}_1 S^{\beta^*/4}} \sum_{(k,l,k_1,l_1) \in \mathbb{V}_2(u)} \Psi(\hat{u}_{k,l,k_1,l_1,\epsilon}) \\
& \leq \mathbb{Q} S^4 e^{-\mathbb{Q}_1 S^{\beta^*/4}} \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \Psi(u_{k,l,\epsilon}^-) \\
& = o(\pi_{1,\epsilon}(u)), \quad u \rightarrow \infty, S \rightarrow \infty.
\end{aligned}$$

Consequently,

$$(41) \quad \Lambda'_2(u) = o(\pi_{1,\epsilon}(u)), \quad u \rightarrow \infty, S \rightarrow \infty.$$

Combing (35), (38), (39) (40) with (41), we derive that

$$\pi_1(u) \sim 4\Gamma(1/\beta_1 + 1)\Gamma(1/\beta_2 + 1) \prod_{i=1}^2 \mathcal{H}_{\alpha_i} \frac{\overleftarrow{v}_1(1/u)}{\overleftarrow{\rho}_1(1/u)} \frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)} \Psi(u), \quad u \rightarrow \infty,$$

hence the claim follows.

Case ii) $\gamma_1 = 0, \gamma_2 \in (0, \infty)$. Let in the sequel

$$\tilde{I}_{k,0}(u) = I_{k,0}(u) \cup I_{k,-1}(u), \quad \mathbb{V}_1^{(1)}(u) = \{(k, k_1) : -N_1(u) - 2 \leq k < k_1 \leq N_1(u) + 1, k_1 - k \geq 2\},$$

and

$$\mathbb{V}_2^{(1)}(u) = \{(k, k_1) : -N_1(u) - 2 \leq k < k_1 \leq N_1(u) + 1, k_1 = k + 1\}.$$

For any $0 < \epsilon < 1$ and all u large enough

$$(42) \quad \pi_{1,\epsilon}^{(1)}(u) - \sum_{i=1}^2 \Lambda_i^{(1)}(u) \leq \pi_1(u) \leq \pi_{1,-\epsilon}^{(1)}(u) + \pi_{1,-\epsilon}^{(2)}(u),$$

with

$$\pi_{1,\pm\epsilon}^{(1)}(u) := \sum_{k=-N_1(u)\pm 2}^{N_1(u)\mp 1} \mathbb{P} \left(\sup_{(s,t) \in I_{k,0}(u)} \frac{\overline{X}(s,t)}{1 + (1 \pm \epsilon)v_2^2(t)} > u_{k,\epsilon}^{1,\pm} \right)$$

$$\begin{aligned}\pi_{1,-\epsilon}^{(2)}(u) &= \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{|l|=1, l \neq -1}^{N_2(u)+1} \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^- \right) \\ \Lambda_i^{(1)}(u) &= \sum_{(k,k_1) \in \mathbb{V}_i^{(1)}(u)} \mathbb{P} \left(\sup_{(s,t) \in \tilde{I}_{k,0}(u)} \overline{X}(s,t) > u_{k,\epsilon}^{1,-}, \sup_{(s,t) \in \tilde{I}_{k_1,0}(u)} \overline{X}(s,t) > u_{k_1,\epsilon}^{1,-} \right), \quad i = 1, 2,\end{aligned}$$

Set further $X_{u,k}(s,t) = \overline{X}(k \overleftarrow{\rho}_1(u^{-1})S + s, t)$ and define

$$\mathcal{K}_u = \{k, |k| \leq N_1(u) + 2\}, \quad \mathcal{E}_u = \tilde{I}_{0,0}(u), \quad h_k(u) = u_{k,\epsilon}^{1,-}, \quad d_u(s,t) = (1-\epsilon)v_2^2(t).$$

Using Lemma 5.2, we have

$$\lim_{u \rightarrow \infty} \sup_{k \in \mathcal{K}_u} \left| (\Psi(u_{k,\epsilon}^{1,-}))^{-1} \mathbb{P} \left(\sup_{(s,t) \in \tilde{I}_{k,0}(u)} \frac{\overline{X}(s,t)}{1 + (1-\epsilon)v_2^2(t)} > u_{k,\epsilon}^{1,-} \right) - \mathcal{H}_{\alpha_1}[0, S] \widehat{\mathcal{P}}_{\alpha_2}^{\gamma_2(1-\epsilon)}(S) \right| = 0.$$

Further, by Lemma 6.2, we have

$$\begin{aligned}\pi_{1,-\epsilon}^{(1)}(u) &\sim \sum_{k=-N_1(u)-2}^{N_1(u)+1} \mathcal{H}_{\alpha_1}[0, S] \widehat{\mathcal{P}}_{\alpha_2}^{\gamma_2(1-\epsilon)}(S) \Psi(u_{k,\epsilon}^{1,-}) \\ &\leq 2\mathcal{H}_{\alpha_1}[0, S] \widehat{\mathcal{P}}_{\alpha_2}^{\gamma_2(1-\epsilon)}(S) \left(2k_\epsilon + 1 + \frac{\Psi(u)}{\overleftarrow{\rho}_1(u^{-1})S} \int_0^{\overleftarrow{v}_1(u^{-1} \ln u)} e^{-(1-\epsilon)^2 u^2 v_1^2(t)} dt \right) (1 + o(1)) \\ (43) \quad &\sim 2\Gamma(1/\beta_1 + 1) \mathcal{H}_{\alpha_1} \widehat{\mathcal{P}}_{\alpha_2}^{\gamma_2} \frac{\overleftarrow{v}_1(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})} \Psi(u), \quad u \rightarrow \infty, \epsilon \rightarrow 0, S \rightarrow \infty.\end{aligned}$$

Similarly, as $u \rightarrow \infty, \epsilon \rightarrow 0, S \rightarrow \infty$,

$$(44) \quad \pi_{1,\epsilon}^{(1)}(u) \sim 2\Gamma(1/\beta_1 + 1) \mathcal{H}_{\alpha_1} \widehat{\mathcal{P}}_{\alpha_2}^{\gamma_2} \frac{\overleftarrow{v}_1(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})} \Psi(u).$$

Moreover, by Lemma 5.2

$$\begin{aligned}\pi_{1,-\epsilon}^{(2)}(u) &\sim \sum_{k=-N_1(u)-2}^{N_1(u)+2} \sum_{|l|=1, l \neq -1}^{N_2(u)+1} \prod_{i=1}^2 \mathcal{H}_{\alpha_i}[0, S] \Psi(u_{k,l,\epsilon}^-) \\ &\leq 2 \prod_{i=1}^2 \mathcal{H}_{\alpha_i}[0, S] \Psi(u) \sum_{k=-N_1(u)-2}^{N_1(u)+2} e^{-(1-\epsilon)u^2 \inf_{s \in I_k(u)} v_1^2(|s|)} \sum_{|l|=1, l \neq -1}^{N_2(u)+1} e^{-(1-\epsilon)u^2 v_2^2(\overleftarrow{\rho}_2(u^{-1}))} |lS|^{\beta'_2} \\ (45) \quad &\leq 2 \prod_{i=1}^2 \mathcal{H}_{\alpha_i}[0, S] \Psi(u) \sum_{k=-N_1(u)-2}^{N_1(u)+1} e^{-(1-\epsilon)u^2 \inf_{s \in I_k(u)} v_1^2(|s|)} \sum_{|l|=1, l \neq -1}^{N_2(u)+1} e^{-(1-2\epsilon)\gamma_2} |lS|^{\beta'_2} \\ &\leq 4 \prod_{i=1}^2 \mathcal{H}_{\alpha_i} \frac{\overleftarrow{v}_1(u^{-1})\Psi(u)}{(1-\epsilon)^{2/\beta_1} \overleftarrow{\rho}_1(u^{-1})} S^2 e^{-Q_4 S^{\beta'_2}} \\ (46) \quad &= o\left(\pi_{1,\epsilon}^{(1)}(u)\right)\end{aligned}$$

as $u \rightarrow \infty, S \rightarrow \infty$ and $\epsilon \rightarrow 0$, with $0 < \beta'_2 < \beta_2$. Note that in (45) we use Potter's bounds (see e.g., [34]) for regularly varying function $v_2(t)$ at zero to derive that, for u and S large enough,

$$(47) \quad \frac{v_2^2(\overleftarrow{\rho}_2(u^{-1})lS)}{v_2^2(\overleftarrow{\rho}_2(u^{-1}))} \geq (lS)^{\beta'_2}$$

holds for $1 \leq l \leq N_2(u)$. Using Lemma 5.4, we have

$$\begin{aligned}\Lambda_1^{(1)}(u) &\leq \mathbb{Q}S^4 \sum_{(k,k_1) \in \mathbb{V}_1^{(1)}(u)} \Psi(u_{k,k_1,\epsilon}^{1,-}) e^{-Q_1(|k-k_1|S)^{\beta^*/2}} \\ &\leq \mathbb{Q}S^4 \sum_{k=-N_1(u)-2}^{N_1(u)+1} \Psi(u_{k,\epsilon}^{1,-}) \sum_{m \geq 1} e^{-Q_1 m^{\beta^*/2} S^{\beta^*/2}}\end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{Q}S^4 \sum_{k=-N_1(u)-2}^{N_1(u)+1} \Psi(u_{k,\epsilon}^{1,-}) e^{-\mathbb{Q}_2 S^{\beta^*}/2} \\
(48) \quad &= o\left(\pi_{1,\epsilon}^{(1)}(u)\right), \quad u \rightarrow \infty, S \rightarrow \infty,
\end{aligned}$$

with $\hat{u}_{k,k_1,\epsilon} = \min(u_{k,\epsilon}^{1,-}, u_{k_1,\epsilon}^{1,-})$ and $\beta^* = \min(\alpha_1, \alpha_2)$. Using Lemma 5.2 yields that

$$\begin{aligned}
(49) \quad \Lambda_2^{(1)}(u) &\leq \sum_{k=-N_1(u)-2}^{N_1(u)+1} \left[\mathbb{P} \left\{ \sup_{(s,t) \in \tilde{I}_{k,0}(u)} \frac{\overline{X}(s,t)}{1 + (1-\epsilon)v_2^2(t)} > u_{k,\epsilon}^{1,-} \right\} \right. \\
&\quad + \mathbb{P} \left\{ \sup_{(s,t) \in \tilde{I}_{k+1,0}(u)} \frac{\overline{X}(s,t)}{1 + (1-\epsilon)v_2^2(t)} > u_{k+1,\epsilon}^{1,-} \right\} \\
&\quad \left. - \mathbb{P} \left\{ \sup_{(s,t) \in \tilde{I}_{k,0}(u) \cup \tilde{I}_{k+1,0}(u)} \frac{\overline{X}(s,t)}{1 + (1+\epsilon)v_2^2(t)} > \check{u}_{k,k+1,\epsilon} \right\} \right] \\
&\leq ((1+\epsilon)2\mathcal{H}_{B_{\alpha_1}}[0, S] - (1-\epsilon)\mathcal{H}_{B_{\alpha_1}}[0, 2S]) \\
&\quad \times \widehat{\mathcal{P}}_{\alpha_2}^{\gamma_2(1-\epsilon)}(S) \sum_{k=-N_1(u)-2}^{N_1(u)+1} \Psi(u_{k,\epsilon}^{1,-}) \\
(50) \quad &= o\left(\pi_{1,\epsilon}^{(1)}(u)\right), \quad u \rightarrow \infty, S \rightarrow \infty,
\end{aligned}$$

with $\check{u}_{k,k_1,\epsilon} = \max(u_{k,\epsilon}^{1,-}, u_{k_1,\epsilon}^{1,-})$. Combining (42), (43), (44), (46), (48) with (49) leads to

$$\pi_1(u) \sim 2\Gamma(1/\beta_1 + 1)\mathcal{H}_{\alpha_1} \widehat{\mathcal{P}}_{\alpha_2}^{\gamma_2} \frac{\overleftarrow{v}_1(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})} \Psi(u), \quad u \rightarrow \infty.$$

The proof is completed by inserting the above asymptotic into (33).

Case ii) $\gamma_1 = 0, \gamma_2 = \infty$. In this case (42), (43), (44), (46), (48) and (49) still hold except for the fact that in light of Lemma 5.2, we have to replace both $\widehat{\mathcal{P}}_{\alpha_2}^{\gamma_2(1-\epsilon)}(S)$ and $\widehat{\mathcal{P}}_{\alpha_2}^{\gamma_2}$ by 1 in (43) and (44).

Case iii) $\gamma_1, \gamma_2 \in (0, \infty)$. Let $\widehat{I}_{0,0}(u) = I_{0,0}(u) \cup I_{-1,0}(u) \cup I_{0,-1}(u) \cup I_{-1,-1}(u)$. It follows straightforwardly that for any $0 < \epsilon < 1/2$ and u large enough

$$(51) \quad \pi_{1,\epsilon}^{(3)}(u) \leq \pi_1(u) \leq \pi_{1,-\epsilon}^{(3)}(u) + \pi_{1,-\epsilon}^{(4)}(u)$$

with

$$\pi_{1,\pm\epsilon}^{(3)}(u) = \mathbb{P} \left(\sup_{(s,t) \in \widehat{I}_{0,0}(u)} \frac{\overline{X}(s,t)}{(1 + (1 \pm \epsilon)v_1^2(s))(1 + (1 \pm \epsilon)v_2^2(t))} > u \right),$$

and

$$\pi_{1,-\epsilon}^{(4)}(u) = \sum_{|k|=1, k \neq -1}^{N_1(u)+2} \sum_{|l|=1, l \neq -1}^{N_2(u)+2} \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^- \right).$$

By Lemma 5.2, it follows that

$$(52) \quad \pi_{1,\pm\epsilon}^{(3)}(u) \sim \prod_{i=1}^2 \widehat{\mathcal{P}}_{\alpha_i}^{(1 \pm \epsilon)\gamma_i}(S) \Psi(u), \quad u \rightarrow \infty, S \rightarrow \infty.$$

In addition, using Lemma 5.2 and (47), the same argument as given in the derivation of the upper bound for $\pi_{1,-\epsilon}^{(2)}(u)$ yields

$$(53) \quad \pi_{1,-\epsilon}^{(4)}(u) = o(\pi_{1,\pm\epsilon}^{(3)}(u))$$

as $u \rightarrow \infty$ and $S \rightarrow \infty$. Combination of (51) and (52) with (53) leads to

$$\pi_1(u) \sim \prod_{i=1}^2 \widehat{\mathcal{P}}_{\alpha_i}^{\gamma_i} \Psi(u), \quad u \rightarrow \infty, S \rightarrow \infty,$$

hence the proof of this case is complete.

Case iii) $\gamma_1 \in (0, \infty), \gamma_2 = \infty$. The proof follows the same lines as given in previous case, with $\widehat{\mathcal{P}}_{\alpha_2}^{\gamma_2}$ replaced by 1.

Case iii) $\gamma_1, \gamma_2 = \infty$. Similarly, (51), (52) and (53) hold with $\widehat{\mathcal{P}}_{\alpha_1}^{\gamma_1}, \widehat{\mathcal{P}}_{\alpha_2}^{\gamma_2}$ replaced by 1. \square

Proof of Theorem 3.7 Similarly as in (35)

$$(54) \quad \pi_1^+(u) - \Lambda^{(1)}(u) \leq \pi_1(u) \leq \pi_1^-(u),$$

with

$$\begin{aligned} \pi_1^\pm(u) : &= \sum_{k=-N_1(u) \mp 2}^{N_1(u) \mp 1} \sum_{l=-N_2(u) \pm 2}^{N_2(u) \mp 1} \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^\pm \right), \\ \Lambda^{(1)}(u) : &= \sum_{(k,l,k_1,l_1) \in \mathbb{V}_1(u) \cup \mathbb{V}_2(u)} \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} X(s,t) > u, \sup_{(s,t) \in I_{k_1,l_1}(u)} X(s,t) > u \right). \end{aligned}$$

Since B is non-singular matrix, then there exists a positive constant $\mu > 0$ such that for any s, t ,

$$|b_{11}s + b_{12}t| + |b_{21}s + b_{22}t| \geq \mu(|s| + |t|).$$

Thus, for $(s, t) \in I_{k,l}(u)$ with $|k| \geq k_0 \geq 2$ and $|l| \geq l_0 \geq 2$

$$|b_{11}s + b_{12}t| + |b_{21}s + b_{22}t| \geq \mu S \left((k_0 - 1) \overleftarrow{p}_1(u^{-1}) + (l_0 - 1) \overleftarrow{p}_2(u^{-1}) \right).$$

By UCT, for any $(s, t), (s', t') \in I_{k,l}(u)$ with $k_0 \leq |k| \leq N_1(u) + 2, l_0 \leq |l| \leq N_2(u) + 2$ and u large enough set $a(s, t) := v_1^2(|b_{11}s + b_{12}t|) + v_2^2(|b_{21}s + b_{22}t|)$

$$\frac{a(s, t)}{v_1^2(|b_{11}s + b_{12}t| + |b_{21}s + b_{22}t|)} \geq (1 - \epsilon/3) (\nu^{\beta_1} + \theta(1 - \nu)^{\beta_1})$$

and

$$\frac{a(s', t')}{v_1^2(|b_{11}s + b_{12}t| + |b_{21}s + b_{22}t|)} \leq \frac{(1 - \epsilon/3)^2}{1 - \epsilon} ((\nu + \delta)^{\beta_1} + \theta(1 - \nu + \delta)^{\beta_1}),$$

where

$$\nu = \frac{|b_{11}s + b_{12}t|}{|b_{11}s + b_{12}t| + |b_{21}s + b_{22}t|} \in [0, 1],$$

and

$$0 \leq \delta \leq \frac{(|b_{11}| + |b_{12}| + |b_{21}| + |b_{22}|) (\overleftarrow{p}_1(u^{-1}) + \overleftarrow{p}_2(u^{-1})) S}{|b_{11}s + b_{12}t| + |b_{21}s + b_{22}t|} \leq \frac{2 \left(\sum_{i,j=1}^2 |b_{ij}| \right) (1 + \eta^{-1/\alpha_1})}{\mu ((k_0 - 1) + (l_0 - 1) \eta^{-1/\alpha_1})} \rightarrow 0,$$

as $k_0, l_0 \rightarrow \infty$. Using that for any $0 < \epsilon < 1/2$, when k_0 and l_0 are large enough

$$(\nu^{\beta_1} + \theta(1 - \nu)^{\beta_1}) \geq (1 - \epsilon/3) ((\nu + \delta)^{\beta_1} + \theta(1 - \nu + \delta)^{\beta_1}), \quad \nu \in [0, 1]$$

for any $0 < \epsilon < 1/2$, there exists k_ϵ, l_ϵ such that for any $k_\epsilon \leq |k| \leq N_1(u) + 2, l_\epsilon \leq |l| \leq N_2(u) + 2$ and u large enough

$$a(s, t) \geq (1 - \epsilon)a(s', t'), \quad (s, t), (s', t') \in I_{k,l}(u),$$

which is equivalent to

$$(55) \quad \inf_{(s,t) \in I_{k,l}(u)} a(s, t) \geq (1 - \epsilon) \sup_{(s,t) \in I_{k,l}(u)} a(s, t).$$

Case i). Using (37) and by (55), we have

$$\begin{aligned} \pi_1^-(u) &\sim \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \prod_{i=1}^2 \mathcal{H}_{\alpha_i}[0, S] \Psi(u_{k,l,\epsilon}^-) \\ &\sim \prod_{i=1}^2 \mathcal{H}_{\alpha_i}[0, S] \Psi(u) \sum_{k=-N_1(u)-2}^{N_1(u)+1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} e^{-(1-\epsilon)u^2 \inf_{(s,t) \in I_{k,l}(u)} a(s,t)} \\ &\leq \prod_{i=1}^2 \mathcal{H}_{\alpha_i}[0, S] \Psi(u) (R_3(u) + R_4(u) + \end{aligned}$$

$$+ \frac{1}{\overleftarrow{\rho}_1(u^{-1})\overleftarrow{\rho}_2(u^{-1})S^2} \sum_{|k|=k_\epsilon}^{N_1(u)+2} \sum_{|l|=l_\epsilon}^{N_2(u)+2} \int_{(s,t) \in I_{k,l}(u)} e^{-(1-\epsilon)^2 u^2 a(s,t)} ds dt \Bigg),$$

where

$$R_3(u) = \sum_{|k| \leq k_\epsilon} \sum_{l=-N_2(u)-2}^{N_2(u)+1} e^{-(1-\epsilon)u^2 \inf_{(s,t) \in I_{k,l}(u)} a(s,t)},$$

and

$$R_4(u) = \sum_{|l| \leq l_\epsilon} \sum_{k=-N_1(u)-2}^{N_1(u)+1} e^{-(1-\epsilon)u^2 \inf_{(s,t) \in I_{k,l}(u)} a(s,t)}.$$

By linear transformation $(s', t')^\top = B(s, t)^\top$ and Lemma 6.2, we have

$$\begin{aligned} & \frac{1}{\overleftarrow{\rho}_1(u^{-1})\overleftarrow{\rho}_2(u^{-1})} \sum_{|k|=k_\epsilon}^{N_1(u)+2} \sum_{|l|=l_\epsilon}^{N_2(u)+2} \int_{(s,t) \in I_{k,l}(u)} e^{-(1-\epsilon)^2 u^2 a(s,t)} ds dt \\ & \leq \frac{1}{\overleftarrow{\rho}_1(u^{-1})\overleftarrow{\rho}_2(u^{-1})} \int_{-2\overleftarrow{b}_1(u^{-1} \ln u)}^{2\overleftarrow{b}_1(u^{-1} \ln u)} \int_{-2\overleftarrow{b}_2(u^{-1} \ln u)}^{2\overleftarrow{b}_2(u^{-1} \ln u)} e^{-(1-\epsilon)^2 u^2 a(s,t)} ds dt \\ & \leq \frac{1}{\overleftarrow{\rho}_1(u^{-1})\overleftarrow{\rho}_2(u^{-1})} \frac{1}{|\det(B)|} \int_{-\mathbb{Q}\overleftarrow{b}_1(u^{-1} \ln u)}^{\mathbb{Q}\overleftarrow{b}_1(u^{-1} \ln u)} \int_{-\mathbb{Q}\overleftarrow{b}_2(u^{-1} \ln u)}^{\mathbb{Q}\overleftarrow{b}_2(u^{-1} \ln u)} e^{-(1-\epsilon)^2 u^2 v_1^2(|s'|)} e^{-(1-\epsilon)^2 u^2 v_2^2(|t'|)} ds' dt' \\ & = \frac{1}{\overleftarrow{\rho}_1(u^{-1})\overleftarrow{\rho}_2(u^{-1})} \frac{4}{|\det(B)|} \int_0^{\mathbb{Q}\overleftarrow{b}_1(u^{-1} \ln u)} \int_0^{\mathbb{Q}\overleftarrow{b}_2(u^{-1} \ln u)} e^{-(1-\epsilon)^2 u^2 v_1^2(|s'|)} e^{-(1-\epsilon)^2 u^2 v_2^2(|t'|)} ds' dt' \\ & \sim (1-\epsilon)^{-1/\beta_1-1/\beta_2} \frac{4}{|\det(B)|} \Gamma(1/\beta_1+1) \Gamma(1/\beta_2+1) \frac{\overleftarrow{v}_1(u^{-1})\overleftarrow{v}_2(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})\overleftarrow{\rho}_2(u^{-1})} \rightarrow \infty, \quad u \rightarrow \infty. \end{aligned}$$

Moreover, in light of Lemma 6.4, there exists a constant $\kappa_1 > 0$ such that

$$(56) \quad \kappa_1 v_1^2(|s|) + \kappa_1 v_2^2(|t|) \leq a(s, t), \quad s, t \in \mathbb{R}.$$

Thus we have

$$\begin{aligned} R_3(u) & \leq \sum_{|k| \leq k_\epsilon} \sum_{l=-N_2(u)-2}^{N_2(u)+1} e^{-(1-\epsilon)u^2 \inf_{(s,t) \in I_{k,l}(u)} \kappa_1(v_1^2(|s|) + v_2^2(|t|))} \\ & \leq (2k_\epsilon + 1) \sum_{l=-N_2(u)-2}^{N_2(u)+1} e^{-(1-\epsilon)u^2 \inf_{t \in J_l(u)} \kappa_1 v_2^2(|t|)} \\ & \leq \mathbb{Q} \frac{\overleftarrow{v}_2(u^{-1})}{\overleftarrow{\rho}_2(u^{-1})} = o\left(\frac{\overleftarrow{v}_1(u^{-1})\overleftarrow{v}_2(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})\overleftarrow{\rho}_2(u^{-1})}\right), \quad u \rightarrow \infty. \end{aligned}$$

Similarly,

$$R_4(u) \leq \mathbb{Q}_1 \frac{\overleftarrow{v}_1(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})} = o\left(\frac{\overleftarrow{v}_1(u^{-1})\overleftarrow{v}_2(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})\overleftarrow{\rho}_2(u^{-1})}\right), \quad u \rightarrow \infty.$$

Therefore,

$$(57) \quad \pi_1^-(u) \leq \frac{4}{|\det(B)|} \prod_{i=1}^2 \left[\Gamma(1/\beta_i + 1) \mathcal{H}_{\alpha_i} \frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)} \right] \Psi(u) (1 + o(1)), \quad u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0.$$

In the same way we obtain that

$$(58) \quad \pi_1^+(u) \geq \frac{4}{|\det(B)|} \prod_{i=1}^2 \left[\Gamma(1/\beta_i + 1) \mathcal{H}_{\alpha_i} \frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)} \right] \Psi(u) (1 + o(1)), \quad u \rightarrow \infty, S \rightarrow \infty.$$

Due to (56), letting

$$(59) \quad Y(s, t) = \frac{\overleftarrow{X}(s, t)}{1 + \frac{\kappa_1}{2} v_1^2(|s|) + \frac{\kappa_1}{2} v_2^2(|t|)}, \quad (s, t) \in \mathbb{R}^2,$$

we have

$$\Lambda^{(1)}(u) \leq \sum_{(k,l,k_1,l_1) \in \mathbb{V}_1(u) \cup \mathbb{V}_2(u)} \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} Y(s,t) > u, \sup_{(s,t) \in I_{k_1,l_1}(u)} Y(s,t) > u \right).$$

The same argument as given in the proof of Theorem 3.1 leads to

$$(60) \quad \Lambda^{(1)}(u) = o(\pi_1^-(u)), \quad u \rightarrow \infty, S \rightarrow \infty.$$

Inserting (57)-(60) into (54) yields

$$\pi_1(u) \sim \frac{4}{|\det(B)|} \prod_{i=1}^2 \left[\Gamma(1/\beta_i + 1) \mathcal{H}_{\alpha_i} \frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)} \right] \Psi(u),$$

which together with (33) completes the proof.

Case ii) $\gamma_1, \gamma_2 \in (0, \infty)$. Using the same notation for $\widehat{I}_{0,0}(u)$ as that in the proof of Theorem 3.1 for case iii) $\gamma_1, \gamma_2 \in (0, \infty)$, (51) holds with

$$\pi_{1,\pm\epsilon}^{(3)}(u) = \mathbb{P} \left(\sup_{(s,t) \in \widehat{I}_{0,0}(u)} \frac{\overline{X}(s,t)}{1 + (1 \pm \epsilon)a(s,t)} > u \right),$$

and

$$\pi_{1,-\epsilon}^{(4)}(u) = \sum_{|k|=1, k \neq -1}^{N_1(u)+2} \sum_{|l|=1, l \neq -1}^{N_2(u)+2} \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} X(s,t) > u \right).$$

Noting that

$$\begin{aligned} & u^2 (v_1^2 (|b_{11} \overleftarrow{\rho}_1(1/u)s + b_{12} \overleftarrow{\rho}_2(1/u)t|) + v_2^2 (|b_{21} \overleftarrow{\rho}_1(1/u)s + b_{22} \overleftarrow{\rho}_2(1/u)t|)) \\ & \rightarrow \gamma_1 |b_{11}s + b_{12}\eta^{-1/\alpha_1}t|^{\alpha_1} + \theta\gamma_1 |b_{21}s + b_{22}\eta^{-1/\alpha_1}t|^{\alpha_1}, \quad u \rightarrow \infty \end{aligned}$$

uniformly with respect to $s, t \in [-S, S]^2$, it follows from Lemma 5.2 that

$$(61) \quad \pi_{1,\pm\epsilon}^{(3)}(u) \sim \widehat{\mathcal{P}}_{\alpha}^{(1 \pm \epsilon)\gamma_1, (1 \pm \epsilon)\theta\gamma_1, B_{\eta, \alpha}}(S) \Psi(u) \sim \widehat{\mathcal{P}}_{\alpha}^{\gamma_1, \theta\gamma_1, B_{\eta, \alpha}} \Psi(u), \quad u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0.$$

Moreover, by Lemma 6.4 and (53), with Y defined by (59),

$$\pi_{1,-\epsilon}^{(4)}(u) \leq \sum_{|k|=1, k \neq -1}^{N_1(u)+2} \sum_{|l|=1, l \neq -1}^{N_2(u)+2} \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} Y(s,t) > u \right) = o(\Psi(u)),$$

as $u \rightarrow \infty, S \rightarrow \infty$. Thus $\pi_1(u) \sim \widehat{\mathcal{P}}_{\alpha}^{\gamma_1, \theta\gamma_1, B_{\eta, \alpha}} \Psi(u)$, which completes the proof.

Case ii) $\gamma_1 = \gamma_2 = \infty$. The proof follows by the same argument as the proof of Case ii) $\gamma_1, \gamma_2 \in (0, \infty)$, with $\widehat{\mathcal{P}}_{\alpha}^{\gamma_1, \theta\gamma_1, B_{\eta, \alpha}}$ replaced by 1. \square

Proof of Theorem 3.3 This scenario requires a modification of set D_u . Let $D_u^{(1)} = \{(s,t), |s+b_{12}t| \leq \overleftarrow{v}_1(u^{-1} \ln u), |t| \leq \overleftarrow{v}_2(u^{-1} \ln u)\}$. It follows that (33)-(34) also hold with D_u replaced by $D_u^{(1)}$. In this scenario, denote $\alpha = \alpha_1 = \alpha_2$.

Case i) $\gamma_1 = \gamma_2 = 0$. Let

$$E_l^+(u) = \{k : I_{k,l}(u) \subset D_u^{(1)}\}, E_l^-(u) = \{k : I_{k,l}(u) \cap D_u^{(1)} \neq \emptyset\},$$

$$E^{(1)}(u) = \{(k,l,k_1,l_1), k \leq k_1, I_{k,l}(u) \cap D_u^{(1)} \neq \emptyset, I_{k_1,l_1}(u) \cap D_u^{(1)} \neq \emptyset \text{ and } I_{k,l}(u) \cap I_{k',l'}(u) = \emptyset\},$$

$$E^{(2)}(u) = \{(k,l,k_1,l_1), k \leq k_1, I_{k,l}(u) \cap D_u^{(1)} \neq \emptyset, I_{k_1,l_1}(u) \cap D_u^{(1)} \neq \emptyset, (k,l) \neq (k_1,l_1) \text{ and } I_{k,l}(u) \cap I_{k',l'}(u) \neq \emptyset\}.$$

It follows that

$$(62) \quad \pi_2^+(u) - \sum_{i=1}^2 \Lambda_i^{(2)}(u) \leq \pi_1(u) \leq \pi_2^-(u),$$

where

$$\pi_2^{\pm}(u) : = \sum_{l=-N_2(u) \mp 1}^{N_2(u) \mp 1} \sum_{k \in E_l^{\pm}(u)} \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^{\pm} \right),$$

$$\Lambda_i^{(2)}(u) : = \sum_{(k,l,k_1,l_1) \in E^{(i)}(u)} \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} \overline{X}(s,t) > u_{k,l,\epsilon}^-, \sup_{(s,t) \in I_{k_1,l_1}(u)} \overline{X}(s,t) > u_{k_1,l_1,\epsilon}^- \right).$$

Using (37), we have

$$\begin{aligned} \pi_2^-(u) &= \sum_{l=-N_2(u)-2}^{N_2(u)+1} \sum_{k \in E_0^-(u)} \mathbb{P} \left(\sup_{(s,t) \in I_{0,0}(u)} \overline{X}(\check{\rho}_1(u^{-1})kS + s, \check{\rho}_2(u^{-1})lS + t) > u_{k,l,\epsilon}^- \right) \\ &\sim \sum_{l=-N_2(u)-2}^{N_2(u)+1} \sum_{k \in E_l^-(u)} \prod_{i=1}^2 \mathcal{H}_{\alpha_i}[0, S] \Psi(u_{k,l,\epsilon}^-) \\ &\sim \prod_{i=1}^2 \mathcal{H}_{\alpha_i}[0, S] \Psi(u) \sum_{l=-N_2(u)-2}^{N_2(u)+1} \sum_{k \in E_l^-(u)} e^{-(1-\epsilon)u^2 \inf_{(s,t) \in I_{k,l}(u)} (v_1^2(|s+b_{12}t|) + v_2^2(|t|))} \end{aligned}$$

We observe that, for u sufficiently large and all $|l| \leq N_2(u) + 2$,

$$E_l^- \subset \left\{ k \in \mathbb{N}, |k - \left[b_{12}l \frac{\overleftarrow{\rho}_2(1/u)}{\overleftarrow{\rho}_1(1/u)} \right]| \leq N_1(u) + 2(2 + \lceil |b_{12}|\eta^{-1/\alpha} \rceil) \right\},$$

and

$$E_l^+ \supset \left\{ k \in \mathbb{N}, |k - \left[b_{12}l \frac{\overleftarrow{\rho}_2(1/u)}{\overleftarrow{\rho}_1(1/u)} \right]| \leq N_1(u) - 2(2 + \lceil |b_{12}|\eta^{-1/\alpha} \rceil) \right\}.$$

By UCT, we have that for any $\epsilon > 0$ there exists $\ell_\epsilon > 0$ such that

$$\inf_{t \in J_l(u)} v_2^2(|t|) \geq (1 - \epsilon) \sup_{t \in J_l(u)} v_2^2(|t|)$$

holds for $\ell_\epsilon \leq |l| \leq N_2(u) + 2$. Moreover, for any $\epsilon > 0$ there exists $k_\epsilon > 0$ such that

$$\inf_{t \in I_{k,l}(u)} v_1^2(|s + b_{12}t|) \geq (1 - \epsilon) \sup_{s \in I_k(u)} v_1^2(|s + b_{12}l \frac{\overleftarrow{\rho}_2(1/u)}{\overleftarrow{\rho}_1(1/u)} S|)$$

hold for $|l| \leq N_2(u) + 2$ and

$$k \in E_{l,\epsilon}^-(u) := \left\{ k, k_\epsilon \leq |k - \left[b_{12}l \frac{\overleftarrow{\rho}_2(1/u)}{\overleftarrow{\rho}_1(1/u)} \right]| \leq N_1(u) + 2(1 + \lceil |b_{12}|\eta^{-1/\alpha} \rceil) \right\}.$$

Therefore, in light of Lemma 6.2, we have

$$\begin{aligned} &\sum_{l=-N_2(u)-2}^{N_2(u)+1} \sum_{k \in E_l^-(u)} e^{-(1-\epsilon)u^2 \inf_{(s,t) \in I_{k,l}(u)} (v_1^2(|s+b_{12}t|) + v_2^2(|t|))} \\ &\leq \sum_{l=-N_2(u)-2}^{N_2(u)+1} e^{-(1-\epsilon)u^2 \inf_{t \in J_l(u)} v_2^2(|t|)} \sum_{k \in E_l^-(u)} e^{-(1-\epsilon)u^2 \inf_{(s,t) \in I_{k,l}(u)} v_1^2(|s+b_{12}t|)} \\ &\leq \frac{1}{\overleftarrow{\rho}_2(u^{-1})S} \sum_{|l| \geq \ell_\epsilon}^{N_2(u)+2} \int_{t \in J_l(u)} e^{-(1-\epsilon)^2 u^2 v_2^2(|t|)} dt \\ &\times \left(2k_\epsilon + 1 + \frac{1}{\overleftarrow{\rho}_1(1/u)S} \sum_{k \in E_{l,\epsilon}(u)} \int_{s \in I_k(u)} e^{-(1-\epsilon)^2 u^2 v_1^2(|s+b_{12}l \frac{\overleftarrow{\rho}_2(1/u)}{\overleftarrow{\rho}_1(1/u)} S|)} ds \right) \\ &+ \sum_{|l|=0}^{\ell_\epsilon} \left(2k_\epsilon + 1 + \frac{1}{\overleftarrow{\rho}_1(1/u)S} \sum_{k \in E_{l,\epsilon}(u)} \int_{s \in I_k(u)} e^{-(1-\epsilon)^2 u^2 v_1^2(|s+b_{12}l \frac{\overleftarrow{\rho}_2(1/u)}{\overleftarrow{\rho}_1(1/u)} S|)} ds \right) \\ &\leq \frac{2(1+o(1))}{\overleftarrow{\rho}_2(u^{-1})S} \int_0^{\mathbb{Q}\check{b}_2(u^{-1} \ln u)} e^{-(1-\epsilon)^2 u^2 v_2^2(|t|)} dt \left(2k_\epsilon + 1 + \frac{2}{\overleftarrow{\rho}_1(1/u)S} \int_0^{\mathbb{Q}\check{b}_1(u^{-1} \ln u)} e^{-(1-\epsilon)^2 u^2 v_1^2(|s|)} ds \right) \\ &\sim (1 - \epsilon)^{-1/\beta_1 - 1/\beta_2} \frac{4}{S^2} \prod_{i=1}^2 \Gamma(1/\beta_i + 1) \frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)}, \quad u \rightarrow \infty. \end{aligned}$$

Consequently,

$$(63) \quad \pi_2^-(u) \leq 4 \prod_{i=1}^2 \left[\Gamma(1/\beta_i + 1) \mathcal{H}_{\alpha_i} \frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)} \right] \Psi(u)(1 + o(1)), \quad u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0.$$

Let $E_{l,\epsilon}^+(u) := \{k, k_\epsilon \leq |k - [bl \frac{\overleftarrow{\rho}_2(1/u)}{\overleftarrow{\rho}_1(1/u)}]| \leq N_1(u) - 2(1 + [b|\eta^{-1/\alpha}])\}$. Similarly,

$$(64) \quad \begin{aligned} \pi_2^+(u) &\geq \prod_{i=1}^2 \mathcal{H}_{\alpha_i}[0, S] \Psi(u) \sum_{l=-N_2(u)+2}^{N_2(u)-1} \sum_{k \in E_{l,\epsilon}^+(u)} e^{-(1-\epsilon)u^2 \sup_{(s,t) \in I_{k,l}(u)} (v_1^2(|s+bt|) + v_2^2(|t|))} \\ &\geq \prod_{i=1}^2 \mathcal{H}_{\alpha_i}[0, S] \Psi(u) \sum_{|l|=k_\epsilon}^{N_2(u)-2} e^{-(1-\epsilon)u^2 \sup_{t \in J_l(u)} v_2^2(|t|)} \sum_{k \in E_{l,\epsilon}^+(u)} e^{-(1-\epsilon)u^2 \sup_{(s,t) \in I_{k,l}(u)} v_1^2(|s+bt|)} \\ &\sim 4 \prod_{i=1}^2 \left[\Gamma(1/\beta_i + 1) \mathcal{H}_{\alpha_i} \frac{\overleftarrow{v}_i(1/u)}{\overleftarrow{\rho}_i(1/u)} \right] \Psi(u), \quad u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0. \end{aligned}$$

Following the same argumentation as given in (40) and (41), we get that $\Lambda_i^{(2)}(u) = o(\pi_2^-(u))$, $i = 1, 2, u \rightarrow \infty$, which together with (63) and (64) completes the proof.

Case ii) $\gamma_2 = 0, \gamma_1 \in (0, \infty)$. We first introduce

$$\begin{aligned} L_{0,l}^*(u) &= \{(s, t), |s + b_{12}t| \leq \overleftarrow{\rho}_1(1/u)S, t \in [l\overleftarrow{\rho}_2(1/u)S, (l+1)\overleftarrow{\rho}_2(1/u)S]\}, \\ L_{k,l}(u) &= \{(s, t), k\overleftarrow{\rho}_1(1/u)S \leq s + b_{12}t \leq (k+1)\overleftarrow{\rho}_1(1/u)S, t \in [l\overleftarrow{\rho}_2(1/u)S, (l+1)\overleftarrow{\rho}_2(1/u)S]\}, \\ u_{k,l,\epsilon,*}^- &= u \left(1 + (1-\epsilon) \inf_{(s,t) \in L_{k,l}(u)} (v_1^2(|s + b_{12}t|) + v_2^2(|t|)) \right) \end{aligned}$$

with $k, l \in \mathbb{Z}$. Then we have

$$(65) \quad \pi_3^+(u) + \sum_{i=1}^2 \Lambda_i^{(3)}(u) \leq \pi_1(u) \leq \pi_3^-(u) + \pi_4(u),$$

where

$$\begin{aligned} \pi_3^\pm(u) &:= \sum_{l=-N_2(u) \pm 2}^{N_2(u) \mp 1} \mathbb{P} \left(\sup_{(s,t) \in L_{0,l}^*(u)} \frac{\overline{X}(s, t)}{1 + (1 \pm \epsilon)v_1^2(|s + b_{12}t|)} > u_{l,\epsilon}^{2,\pm} \right), \\ \pi_4(u) &:= \sum_{|k| \leq N_1(u)+2, k \neq 0, -1} \sum_{l=-N_2(u) \pm 2}^{N_2(u) \mp 1} \mathbb{P} \left(\sup_{(s,t) \in L_{k,l}(u)} \overline{X}(s, t) > u_{k,l,\epsilon,*}^- \right), \\ \Lambda_1^{(3)}(u) &:= \sum_{-N_2(u)-2 \leq l+1 < l_1 \leq N_2(u)+2} \mathbb{P} \left(\sup_{(s,t) \in L_{0,l}(u)} \overline{X}(s, t) > u_{l,\epsilon}^{2,-}, \sup_{(s,t) \in L_{0,l_1}(u)} \overline{X}(s, t) > u_{l_1,\epsilon}^{2,-} \right) \\ \Lambda_2^{(3)}(u) &:= \sum_{l=-N_2(u)-2}^{N_2(u)+2} \mathbb{P} \left(\sup_{(s,t) \in L_{0,l}(u)} \frac{\overline{X}(s, t)}{1 + (1-\epsilon)v_1^2(|s + b_{12}t|)} > u_{l,\epsilon}^{2,-}, \sup_{(s,t) \in L_{0,l+1}(u)} \frac{\overline{X}(s, t)}{1 + (1-\epsilon)v_1^2(|s + b_{12}t|)} > u_{l+1,\epsilon}^{2,-} \right). \end{aligned}$$

Let

$$\begin{aligned} X_l(s, t) &= \overline{X}(-b_{12}l\overleftarrow{\rho}_2(u^{-1})S + s, l\overleftarrow{\rho}_2(u^{-1})S + t), \quad \mathcal{K}_u = \{l, |l| \leq N_2(u) + 2\}, \quad \mathcal{E}_u = L_{0,0}^*(u), \\ h_l(u) &= u_{l,\epsilon}^{2,-}, \quad d_u(s, t) = (1-\epsilon)v_1^2(|s + b_{12}t|) \end{aligned}$$

Since

$$\lim_{u \rightarrow \infty} \sup_{l \in \mathcal{K}_u} \left| (u_{l,\epsilon}^{2,-})^2 v_1^2(|\overleftarrow{\rho}_1(1/u)s + b_{12}\overleftarrow{\rho}_2(1/u)t|) - \gamma_1|s + b_{12}\eta^{-1/\alpha}t| \right| = 0,$$

uniformly over any compact set, by Lemma 5.2, we have

$$\lim_{u \rightarrow \infty} \sup_{l \in \mathcal{K}_u} \left| (\Psi(u_{l,\epsilon}^{2,-}))^{-1} \mathbb{P} \left(\sup_{(s,t) \in L_{0,0}^*(u)} \frac{\overline{X}(-b_{12}l\overleftarrow{\rho}_2(1/u)S + s, l\overleftarrow{\rho}_2(1/u)S + t)}{1 + (1-\epsilon)v_1^2(|s + b_{12}t|)} > u_{l,\epsilon}^{2,-} \right) - \mathcal{H}_\alpha^{(1-\epsilon)\gamma_1, b_{12}\eta^{-1/\alpha}}(S) \right| = 0.$$

Thus we have,

$$\begin{aligned}
\pi_3^-(u) &= \sum_{l=-N_2(u)-2}^{N_2(u)+1} \mathbb{P} \left(\sup_{(s,t) \in L_{0,0}^*(u)} \frac{\overline{X}(-b_{12}l \overleftarrow{\rho}_2(1/u)S + s, l \overleftarrow{\rho}_2(1/u)S + t)}{1 + (1-\epsilon)v_1^2(|s + b_{12}t|)} > u_{l,\epsilon}^{2,-} \right) \\
&\sim \sum_{l=-N_2(u)-2}^{N_2(u)+1} \Psi(u_{l,\epsilon}^{2,-}) \mathcal{H}_\alpha^{(1-\epsilon)\gamma_1, b_{12}\eta^{-1/\alpha}}(S) \\
&\sim \mathcal{H}_\alpha^{(1-\epsilon)\gamma_1, b_{12}\eta^{-1/\alpha}}(S) \Psi(u) \sum_{l=-N_2(u)-2}^{N_2(u)+1} e^{-(1-\epsilon)u^2 \inf_{t \in J_l(u)} v_2^2(|t|)} \\
(66) \quad &\sim \frac{\mathcal{H}_\alpha^{\gamma_1, b_{12}\eta^{-1/\alpha}}(S)}{S} 2\Gamma(1/\beta_2 + 1) \Psi(u) \frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)} (1 + o(1)), \quad u \rightarrow \infty, v, \epsilon \rightarrow 0.
\end{aligned}$$

Moreover, in light of [7],

$$\lim_{S \rightarrow \infty} \frac{\mathcal{H}_\alpha^{\gamma_1, b_{12}\eta^{-1/\alpha}}(S)}{S} = \mathcal{H}_\alpha^{\gamma_1, b_{12}\eta^{-1/\alpha}} \in (0, \infty).$$

Thus we have

$$(67) \quad \pi_3^-(u) \leq \mathcal{H}_\alpha^{\gamma_1, b_{12}\eta^{-1/\alpha}} 2\Gamma(1/\beta_2 + 1) \Psi(u) \frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)} (1 + o(1)), \quad u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0.$$

Similarly,

$$(68) \quad \pi_3^+(u) \geq \mathcal{H}_\alpha^{\gamma_1, b_{12}\eta^{-1/\alpha}} 2\Gamma(1/\beta_2 + 1) \Psi(u) \frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)} (1 + o(1)), \quad u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0.$$

Note that for u sufficiently large

$$(69) \quad L_{0,0}(u) \subset [-(1 + 2|b_{12}|\eta^{-1/\alpha})\overleftarrow{\rho}_1(1/u)S, (1 + 2|b_{12}|\eta^{-1/\alpha})\overleftarrow{\rho}_1(1/u)S] \times [0, \overleftarrow{\rho}_2(1/u)S] =: J_{0,0}(u).$$

Thus, with $S_2 = (1 + 2|b|\eta^{-1/\alpha})S$, by (37) with $u_{k,l,\epsilon,*}^-$ instead of $u_{k,l,\epsilon}^-$, we obtain

$$\begin{aligned}
\pi_4(u) &= \sum_{|k| \leq N_1(u)+2, k \neq 0, -1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \mathbb{P} \left(\sup_{(s,t) \in L_{0,0}(u)} \overline{X}(k \overleftarrow{\rho}_1(1/u)S - b_{12}l \overleftarrow{\rho}_2(1/u)S + s, l \overleftarrow{\rho}_2(1/u)S + t) > u_{k,l,\epsilon,*}^- \right) \\
&\leq \sum_{|k| \leq N_1(u)+2, k \neq 0, -1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \mathbb{P} \left(\sup_{(s,t) \in J_{0,0}(u)} \overline{X}(k \overleftarrow{\rho}_1(1/u)S - b_{12}l \overleftarrow{\rho}_2(1/u)S + s, l \overleftarrow{\rho}_2(1/u)S + t) > u_{k,l,\epsilon,*}^- \right) \\
&\sim \sum_{|k| \leq N_1(u)+2, k \neq 0, -1} \sum_{l=-N_2(u)-2}^{N_2(u)+1} \widehat{\mathcal{H}}_\alpha(S_2) \mathcal{H}_\alpha(S) \Psi(u) e^{-(1-\epsilon)u^2 \inf_{(s,t) \in L_{k,l}(u)} (v_1^2(|s+bt|) + v_2^2(|t|))} \\
&\leq \widehat{\mathcal{H}}_\alpha(S_2) \mathcal{H}_\alpha(S) \Psi(u) \sum_{1 \leq |k| \leq N_1(u)+2} e^{-\mathbb{Q}u^2 v_1^2(\overleftarrow{\rho}_1(1/u)|k|S)} \sum_{l=-N_2(u)-2}^{N_2(u)+1} e^{-(1-\epsilon)u^2 \inf_{t \in J_l(u)} v_2^2(|t|)} \\
&\leq 2\Gamma(1/\beta_2 + 1)(1-\epsilon)^{-2/\beta_2} \frac{\widehat{\mathcal{H}}_\alpha(S_2)}{S} \frac{\mathcal{H}_\alpha(S)}{S} \frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)} \Psi(u) S \sum_{1 \leq |k| \leq N_1(u)+2} e^{-\mathbb{Q}_1|k|S^{\beta_1/2}} \\
(70) \quad &\leq \mathbb{Q}_2 \frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)} \Psi(u) e^{-\mathbb{Q}_3 S^{\beta_1/2}} = o(\pi_3^-(u)), \quad u \rightarrow \infty, S \rightarrow \infty.
\end{aligned}$$

Following the same idea as given in the proof of Theorem 3.1, we get that $\Lambda_1^{(3)}(u) + \Lambda_2^{(3)}(u) = o(\pi_3^-(u))$, as $u \rightarrow \infty$, which completes the proof of this case.

Case ii) $\gamma_2 = 0, \gamma_1 = \infty$. It follows straightforwardly that, for any $x > 0$ and u sufficiently large,

$$(71) \quad \mathbb{P} \left(\sup_{|t| \leq \overleftarrow{v}_2(u^{-1} \ln u)} X(-b_{12}t, t) > u \right) \leq \pi_1(u) \leq \mathbb{P} \left(\sup_{(s,t) \in D_u^{(1)}} \frac{\overline{X}(s, t)}{1 + x\rho_1^2(|s + b_{12}t|) + v_2^2(|t|)} > u \right).$$

Using that the Gaussian random field on the right hand side of the above satisfies case $\gamma_2 = 0, \gamma_1 = x \in (0, \infty)$, by (66) and (70) we get for S sufficiently large

$$\mathbb{P} \left(\sup_{(s,t) \in D_u^{(1)}} \frac{\bar{X}(s,t)}{1 + x\rho_1^2(|s + b_{12}t|) + v_2^2(|t|)} > u \right) \leq \frac{\mathcal{H}_\alpha^{x, b_{12}\eta^{-1/\alpha}}(S) 2\Gamma(1/\beta_2 + 1) \Psi(u) \frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)}}{S} (1 + o(1)).$$

It follows that for any S positive

$$\begin{aligned} \lim_{x \rightarrow \infty} \mathcal{H}_\alpha^{x, b_{12}\eta^{-1/\alpha}}(S) &= \lim_{x \rightarrow \infty} \mathbb{E} \left\{ \sup_{(s+b_{12}\eta^{-1/\alpha}t, t) \in [-S, S] \times [0, S]} e^{W(s,t) - x|s+b_{12}\eta^{-1/\alpha}t|^\alpha} \right\} \\ &= \mathbb{E} \left\{ \sup_{(s+b_{12}\eta^{-1/\alpha}t, t) \in \{0\} \times [0, S]} e^{W(s,t)} \right\} \\ &= \mathcal{H}_\alpha(|b_{12}|^\alpha \eta^{-1} + 1)^{1/\alpha} S. \end{aligned}$$

Hence, as $u \rightarrow \infty, x \rightarrow \infty, S \rightarrow \infty$

$$\mathbb{P} \left(\sup_{(s,t) \in D_u^{(1)}} \frac{\bar{X}(s,t)}{1 + x\rho_1^2(|s + b_{12}t|) + v_2^2(|t|)} > u \right) \leq 2 (|b_{12}|^\alpha \eta^{-1} + 1)^{1/\alpha} \mathcal{H}_\alpha \Gamma(1/\beta_2 + 1) \Psi(u) \frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)} (1 + o(1)).$$

Further, for the process $X(-b_{12}t, t)$, we have

$$(72) \quad \begin{aligned} 1 - \sqrt{\text{Var}(X(-b_{12}t, t))} &\sim v_2^2(|t|), \quad t \rightarrow 0, \\ 1 - \text{Corr}(X(-b_{12}t, t), X(-b_{12}s, s)) &\sim (|b_{12}|^\alpha \eta^{-1} + 1) \rho_2^2(|t - s|), \quad s, t \rightarrow 0. \end{aligned}$$

Thus in light of Theorem 2.1, we have

$$\mathbb{P} \left(\sup_{|t| \leq \overleftarrow{v}_2(u^{-1} \ln u)} X(-b_{12}t, t) > u \right) \sim 2 (|b_{12}|^\alpha \eta^{-1} + 1)^{1/\alpha} \mathcal{H}_\alpha \Gamma(1/\beta_2 + 1) \Psi(u) \frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)}.$$

Consequently,

$$\pi_1(u) \sim 2 (|b_{12}|^\alpha \eta^{-1} + 1)^{1/\alpha} \mathcal{H}_\alpha \Gamma(1/\beta_2 + 1) \Psi(u) \frac{\overleftarrow{v}_2(1/u)}{\overleftarrow{\rho}_2(1/u)},$$

which completes the proof.

Case iii) $\gamma_2 \in (0, \infty), \gamma_1 = \infty$. Let $\hat{I}_{0,0}^*(u) = \{(s, t), |s + b_{12}t| \leq \overleftarrow{\rho}_1(1/u)S, |t| \leq \overleftarrow{\rho}_2(1/u)S\}$. Then for u sufficiently large, we have

$$(73) \quad \mathbb{P} \left(\sup_{(s,t) \in D_u^{(1)}} X(-b_{12}t, t) > u \right) \leq \pi_1(u) \leq \pi_{1,-\epsilon}^{(5)}(u) + \pi_{1,-\epsilon}^{(6)}(u)$$

with

$$\pi_{1,-\epsilon}^{(5)}(u) = \mathbb{P} \left(\sup_{(s,t) \in \hat{I}_{0,0}^*(u)} \frac{\bar{X}(s,t)}{1 + x\rho_1^2(|s + b_{12}t|) + (1-\epsilon)v_2^2(|t|)} > u \right),$$

and

$$\pi_{1,-\epsilon}^{(6)}(u) = \sum_{|k|=1, k \neq -1}^{N_1(u)+2} \sum_{|l|=1, l \neq -1}^{N_2(u)+2} \mathbb{P} \left(\sup_{(s,t) \in L_{k,l}(u)} \bar{X}(s,t) > u_{k,l,\epsilon}^- \right).$$

Since

$$u^2(x\rho_1^2(|\overleftarrow{\rho}_1(1/u)s + b_{12}\overleftarrow{\rho}_2(1/u)t|) + (1-\epsilon)v_2^2(|\overleftarrow{\rho}_2(1/u)t|)) \rightarrow x|s + b_{12}\eta^{-1/\alpha}t|^\alpha + (1-\epsilon)\gamma_2|t|^\alpha, \quad u \rightarrow \infty$$

uniformly on any compact set, then, by Lemma 5.2,

$$\pi_{1,-\epsilon}^{(5)}(u) \sim \hat{\mathcal{H}}_\alpha^{x, \gamma_2, b_{12}\eta^{-1/\alpha}}(S) \Psi(u), \quad u \rightarrow \infty, \epsilon \rightarrow 0.$$

Moreover, by the same argument as given in case ii), we have

$$\lim_{x \rightarrow \infty} \hat{\mathcal{H}}_\alpha^{x, \gamma_2, b_{12}\eta^{-1/\alpha}}(S) = \hat{\mathcal{P}}_\alpha^{\gamma_2(|b_{12}|^\alpha \eta^{-1} + 1)^{-1}} \left((|b|^\alpha \eta^{-1} + 1)^{1/\alpha} S \right).$$

Then

$$\pi_{1,-\epsilon}^{(5)}(u) \sim \hat{\mathcal{P}}_\alpha^{\gamma_2(|b_{12}|^\alpha \eta^{-1} + 1)^{-1}} \Psi(u), \quad u \rightarrow \infty, x \rightarrow \infty, \epsilon \rightarrow 0, S \rightarrow \infty.$$

Using that $L_{0,0}(u) \subset J_{0,0}(u)$, with $J_{0,0}(u)$ defined by (69), and following the same steps as in (70), we get

$$\pi_{1,-\epsilon}^{(6)}(u) = o(\Psi(u)), \quad u \rightarrow \infty, S \rightarrow \infty.$$

Hence, from Theorem 2.1 and (72)

$$\mathbb{P} \left(\sup_{(s,t) \in D_u^{(1)}} X(-b_{12}t, t) > u \right) \sim \widehat{\mathcal{P}}_\alpha^{\gamma_2(|b_{12}|^\alpha \eta^{-1} + 1)^{-1}} \Psi(u), \quad u \rightarrow \infty,$$

which establishes the claim.

Case iv) $\gamma_2 = \infty, \gamma_1 = \infty$. Clearly, (73) holds with

$$\pi_{1,-\epsilon}^{(5)}(u) := \mathbb{P} \left(\sup_{(s,t) \in \tilde{I}_{0,0}(u)} \frac{\overline{X}(s, t)}{1 + x\rho_1^2(|s + b_{12}t|) + y\rho_2^2(|t|)} > u \right), \quad x, y > 0.$$

Moreover,

$$\pi_{1,-\epsilon}^{(5)}(u) \sim \widehat{\mathcal{H}}_\alpha^{x,y,b_{12}\eta^{-1/\alpha}}(S) \Psi(u)$$

and

$$\lim_{y \rightarrow \infty} \lim_{x \rightarrow \infty} \widehat{\mathcal{H}}_\alpha^{x,y,b_{12}\eta^{-1/\alpha}}(S) = \lim_{y \rightarrow \infty} \widehat{\mathcal{P}}_\alpha^{y(|b_{12}|^\alpha \eta^{-1} + 1)^{-1}} \left((|b_{12}|^\alpha \eta^{-1} + 1)^{1/\alpha} S \right) = 1.$$

Hence $\pi_{1,-\epsilon}^{(5)}(u) \sim \Psi(u)$, $u \rightarrow \infty, x \rightarrow \infty, y \rightarrow \infty$. The rest of the proof is the same as the case $\gamma_2 \in (0, \infty), \gamma_1 = \infty$. \square

Proof of Theorem 3.5 . We focus on $\pi_1(u)$ as $u \rightarrow \infty$.

Case i) The proof of this case follows line by line the same arguments as given in the proof of Case i) of Theorem 3.9.

Case ii) $\gamma_1 = 0, \gamma_2 \in (0, \infty)$. First we introduce some new notation. Let

$$u_{k,\epsilon}^{*-} = 1 + (1-3\epsilon) \inf_{t \in I_k(u)} (v_1^2(|(1+b_{12}\mu)s|) + v_2^2(|\mu s|)),$$

and

$$\widehat{I}_{k,0}(u) = I_k(u) \times (J_{-1}(u) \cup J_0(u)), \quad v(s, t) = v_1^2(|s + b_{12}t|) + v_2^2(|t|) - v_1^2(|(1+b_{12}\mu)s|) - v_2^2(|\mu s|), \quad (s, t) \in D_u,$$

where μ is defined right before Theorem 3.5. For any $0 < x < y < \frac{S}{2|b_{12}|}$ and $0 < \epsilon < 1/4$, we have

$$(74) \quad \pi_5^+(u) + \Lambda(u) \leq \pi_1(u) \leq \pi_5^-(u) + \pi_6(u) + \pi_7(u) + \pi_8(u),$$

where

$$\begin{aligned} \pi_5^\pm(u) &:= \sum_{k \in E_{x,y}^\pm(u)} \mathbb{P} \left(\sup_{(s,t) \in \widehat{I}_{k,0}(u)} \frac{\overline{X}(s, t)}{1 + (1 \pm \epsilon) (v_1^2(|s + b_{12}t|) + v_2^2(|t|))} > u \right), \\ \pi_6(u) &:= \sum_{k \in E_{0,x}(u)} \mathbb{P} \left(\sup_{(s,t) \in \tilde{I}_{k,0}(u)} \frac{\overline{X}(s, t)}{1 + (1 - \epsilon) (v_1^2(|s + b_{12}t|) + v_2^2(|t|))} > u \right), \\ \pi_7(u) &:= \sum_{k \in E_{y,\infty}(u)} \mathbb{P} \left(\sup_{(s,t) \in \tilde{I}_{k,0}(u)} \frac{\overline{X}(s, t)}{1 + (1 - \epsilon) (v_1^2(|s + b_{12}t|) + v_2^2(|t|))} > u \right), \\ \pi_8(u) &:= \sum_{|k|=0}^{N_1(u)+2} \sum_{|l|=1, l \neq -1}^{N_2(u)+2} \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} \frac{\overline{X}(s, t)}{1 + (1 - \epsilon) (v_1^2(|s + b_{12}t|) + v_2^2(|t|))} > u \right), \\ \Lambda(u) &:= \sum_{k < k_1 \in E_{x,y}^-(u)} \mathbb{P} \left(\sup_{(s,t) \in \tilde{I}_{k,0}(u)} X(s, t) > u, \sup_{(s,t) \in \tilde{I}_{k_1,0}(u)} X(s, t) > u \right), \end{aligned}$$

with

$$\begin{aligned} E_{0,x} &= \{k, |k| \leq N_1(u) + 2, I_k(u) \cap [-\widehat{\rho}_2(u^{-1})x, \widehat{\rho}_2(u^{-1})x] \neq \emptyset\}, \\ E_{x,y}^- &= \{k, |k| \leq N_1(u) + 2, I_k(u) \cap [(-\widehat{\rho}_2(u^{-1})y, -\widehat{\rho}_2(u^{-1})x] \cup [\widehat{\rho}_1(u^{-1})x, \widehat{\rho}_1(u^{-1})y] \neq \emptyset\}, \\ E_{x,y}^+ &= \{k, |k| \leq N_1(u) + 2, I_k(u) \subset [(-\widehat{\rho}_2(u^{-1})y, -\widehat{\rho}_2(u^{-1})x] \cup [\widehat{\rho}_1(u^{-1})x, \widehat{\rho}_1(u^{-1})y]\}, \\ E_{y,\infty}^- &= \{k, |k| \leq N_1(u) + 2, I_k(u) \cap [-\infty, -\widehat{\rho}_2(u^{-1})y] \cup [\widehat{\rho}_1(u^{-1})y, \infty] \neq \emptyset\}. \end{aligned}$$

We observe that for $|s| \in [\frac{i-1}{n}\overleftarrow{\rho}_2(u^{-1}), \frac{i+2}{n}\overleftarrow{\rho}_2(u^{-1})]$ with $x/2 \leq \frac{i}{n} \leq 2y$ and $|t| \in [0, \overleftarrow{\nu}_2(\ln u/u)]$

$$(75) \quad \begin{aligned} & 1 + (1 - \epsilon) (v_1^2(|s + b_{12}t|) + v_2^2(|t|)) \\ & \geq [1 + (1 - 3\epsilon) (v_1^2(|(1 + b_{12}\mu)s|) + v_2^2(|\mu s|))] [1 + (1 - 3\epsilon)v(i\overleftarrow{\rho}_2(u^{-1})/n, t)], \end{aligned}$$

whose proofs is postponed in the Appendix. Let

$$\begin{aligned} X_{u,k}(s, t) &= \overline{X}(k\overleftarrow{\rho}_1(1/u)S + s, t), \quad \mathcal{K}_u = E_{i/n, (i+1)/n}^-, \quad \mathcal{E}_u = \widehat{I}_{0,0}(u), \\ d_u(s, t) &= (1 - 3\epsilon)u^2v(i\overleftarrow{\rho}_2(u^{-1})/n, t), \quad h_k(u) = u_{k,\epsilon}^{*-}. \end{aligned}$$

We note that

$$\lim_{u \rightarrow \infty} \sup_{k \in \mathcal{K}_u, t \in [-S, S]} \left| (u_{k,\epsilon}^{*-})^2 v(i\overleftarrow{\rho}_2(u^{-1})/n, \overleftarrow{\rho}_2(u^{-1})t) - g_{i/n}(t) \right| = 0.$$

Thus in light of Lemma 5.2, we have

$$\lim_{u \rightarrow \infty} \sup_{x/2 \leq \frac{i}{n} \leq 2y} \sup_{k \in E_{i/n, (i+1)/n}^-} \left| (\Psi(u_{k,\epsilon}^{*-}))^{-1} \mathbb{P} \left(\sup_{(s,t) \in \widehat{I}_{0,0}(u)} \frac{\overline{X}(k\overleftarrow{\rho}_1(1/u)S + s, t)}{1 + (1 - 3\epsilon)v(i\overleftarrow{\rho}_2(u^{-1})/n, t)} > u_{k,\epsilon}^{*-} \right) - \mathcal{H}_{\alpha_1}(S) \widehat{\mathcal{P}}_\beta^{(1-3\epsilon)g_{i/n}}(S) \right| = 0.$$

Thus for $[nx] - 1 \leq i \leq [ny]$, it follows that

$$\begin{aligned} & \sum_{k \in E_{i/n, (i+1)/n}^-} \mathbb{P} \left(\sup_{(s,t) \in \widehat{I}_{k,0}(u)} \frac{\overline{X}(s, t)}{1 + (1 - \epsilon) (v_1^2(|s + b_{12}t|) + v_2^2(|t|))} > u \right) \\ & \leq \sum_{k \in E_{i/n, (i+1)/n}^-} \mathbb{P} \left(\sup_{(s,t) \in \widehat{I}_{k,0}(u)} \frac{\overline{X}(s, t)}{[1 + (1 - 3\epsilon) (v_1^2(|(1 + b_{12}\mu)s|) + v_2^2(|\mu s|))] [1 + (1 - 3\epsilon)v(i\overleftarrow{\rho}_2(u^{-1})/n, t)]} > u \right) \\ & \leq \sum_{k \in E_{i/n, (i+1)/n}^-} \mathbb{P} \left(\sup_{(s,t) \in \widehat{I}_{k,0}(u)} \frac{\overline{X}(s, t)}{1 + (1 - 3\epsilon)v(i\overleftarrow{\rho}_2(u^{-1})/n, t)} > u_{k,\epsilon}^{*-} \right) \\ & = \sum_{k \in E_{i/n, (i+1)/n}^-} \mathbb{P} \left(\sup_{(s,t) \in \widehat{I}_{0,0}(u)} \frac{\overline{X}(k\overleftarrow{\rho}_1(1/u)S + s, t)}{1 + (1 - 3\epsilon)v(i\overleftarrow{\rho}_2(u^{-1})/n, t)} > u_{k,\epsilon}^{*-} \right) \\ & \sim \sum_{k \in E_{i/n, (i+1)/n}^-} \mathcal{H}_{\alpha_1}(S) \widehat{\mathcal{P}}_\beta^{(1-3\epsilon)g_{i/n}}(S) \Psi(u_{k,\epsilon}^{*-}) \\ & \sim \mathcal{H}_{\alpha_1}(S) \widehat{\mathcal{P}}_\beta^{(1-3\epsilon)g_{i/n}}(S) \Psi(u) \sum_{k \in E_{i/n, (i+1)/n}^-} e^{-u^2(1-3\epsilon) \inf_{t \in I_k(u)} (v_1^2(|(1+b_{12}\mu)s|) + v_2^2(|\mu s|))} \\ & \leq \frac{\mathcal{H}_{\alpha_1}(S)}{S} \widehat{\mathcal{P}}_\beta^{(1-3\epsilon)g_{i/n}}(S) \frac{\Psi(u)}{\overleftarrow{\rho}_1(u^{-1})} 2 \int_{i\overleftarrow{\rho}_2(u^{-1})/n}^{(i+1)\overleftarrow{\rho}_2(u^{-1})/n} e^{-(1-4\epsilon)\frac{\gamma_2 M_\beta}{\theta} u^2 \rho_2^2(|s|)} ds (1 + o(1)), \end{aligned}$$

as $u \rightarrow \infty$, with M_β defined right before Theorem 3.5. Using the same arguments as in the proof of Lemma 6.2, we have

$$\begin{aligned} \int_{i\overleftarrow{\rho}_2(u^{-1})/n}^{(i+1)\overleftarrow{\rho}_2(u^{-1})/n} e^{-(1-4\epsilon)\frac{\gamma_2 M_\beta}{\theta} u^2 \rho_2^2(|s|)} ds & \sim \frac{2}{\beta} \overleftarrow{\rho}_2(u^{-1}) \int_{(i/n)^{\beta/2}}^{((i+1)/n)^{\beta/2}} t^{2/\beta-1} e^{-(1-4\epsilon)\frac{\gamma_2 M_\beta}{\theta} t^2} dt \\ & \sim \overleftarrow{\rho}_2(u^{-1}) \int_{i/n}^{(i+1)/n} e^{-(1-4\epsilon)\frac{\gamma_2 M_\beta}{\theta} t^\beta} dt. \end{aligned}$$

Hence

$$\begin{aligned} & \sum_{k \in E_{i/n, (i+1)/n}^-} \mathbb{P} \left(\sup_{(s,t) \in \widehat{I}_{k,0}(u)} \frac{\overline{X}(s, t)}{1 + (1 - \epsilon) (v_1^2(|s + b_{12}t|) + v_2^2(|t|))} > u \right) \\ & \leq 2 \frac{\mathcal{H}_{\alpha_1}(S)}{S} \widehat{\mathcal{P}}_\beta^{(1-3\epsilon)g_{i/n}}(S) \frac{\overleftarrow{\rho}_2(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})} \Psi(u) \int_{i/n}^{(i+1)/n} e^{-(1-4\epsilon)\frac{\gamma_2 M_\beta}{\theta} t^\beta} dt (1 + o(1)) \\ & \leq 2 \mathcal{H}_{\alpha_1} \widehat{\mathcal{P}}_\beta^{g_{i/n}} \frac{\overleftarrow{\rho}_2(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})} \Psi(u) \int_{i/n}^{(i+1)/n} e^{-\frac{\gamma_2 M_\beta}{\theta} t^\beta} dt (1 + o(1)), \quad u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0. \end{aligned}$$

Further, by the continuity of $\widehat{\mathcal{P}}_\beta^{g_s}$ over $s \in [x/2, 2y]$, we have

$$\begin{aligned} \pi_5^-(u) &\leq 2\mathcal{H}_{\alpha_1} \frac{\overleftarrow{\rho}_2(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})} \Psi(u) \sum_{i=[nx]-1}^{[ny]+1} \int_{i/n}^{(i+1)/n} \widehat{\mathcal{P}}_\beta^{g_{i/n}} e^{-\frac{\gamma_2 M_\beta}{\theta} t^\beta} dt (1+o(1)) \\ (76) \quad &\leq 2\mathcal{H}_{\alpha_1} \frac{\overleftarrow{\rho}_2(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})} \Psi(u) \int_x^y \widehat{\mathcal{P}}_\beta^{g_t} e^{-\frac{\gamma_2 M_\beta}{\theta} t^\beta} dt (1+o(1)), \quad u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0, n \rightarrow \infty. \end{aligned}$$

Similarly,

$$(77) \quad \pi_5^+(u) \geq 2\mathcal{H}_{\alpha_1} \frac{\overleftarrow{\rho}_2(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})} \Psi(u) \int_x^y \widehat{\mathcal{P}}_\beta^{g_t} e^{-\frac{\gamma_2 M_\beta}{\theta} t^\beta} dt (1+o(1)), \quad u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0, n \rightarrow \infty.$$

Next we focus on $\pi_6(u)$. In light of (56) and (59), we have

$$\pi_6(u) \leq \sum_{k \in E_{0,x}(u)} \mathbb{P} \left(\sup_{(s,t) \in \widehat{I}_{k,0}(u)} Y(s,t) > u \right).$$

Hence, following case ii) $\gamma_1 = 0, \gamma_2 \in (0, \infty)$ in Theorem 3.1, we have

$$\begin{aligned} \pi_6(u) &\leq 2\mathcal{H}_{\alpha_1}(S) \widehat{\mathcal{P}}_\beta^{\gamma_2 \kappa_1/2}(S) \frac{\Psi(u)}{\overleftarrow{\rho}_1(u^{-1})S} \int_0^{x \overleftarrow{\rho}_2(u^{-1})} e^{-\frac{\kappa_1}{2} u^2 v_1^2(t)} dt (1+o(1)) \\ (78) \quad &\leq 2\mathcal{H}_{\alpha_1} \widehat{\mathcal{P}}_\beta^{\gamma_2 \kappa_1/2} \frac{\overleftarrow{\rho}_2(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})} \Psi(u) \int_0^x e^{-\frac{\kappa_1}{2} \frac{\gamma_2}{\theta} t^\beta} dt (1+o(1)), \quad u \rightarrow \infty, S \rightarrow \infty. \end{aligned}$$

Similarly,

$$(79) \quad \pi_7(u) \leq 2\mathcal{H}_{\alpha_1} \widehat{\mathcal{P}}_\beta^{\gamma_2 \kappa_1/2} \frac{\overleftarrow{\rho}_2(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})} \Psi(u) \int_y^\infty e^{-\frac{\kappa_1}{2} \frac{\gamma_2}{\theta} t^\beta} dt (1+o(1)), \quad u \rightarrow \infty, S \rightarrow \infty.$$

Moreover, by Lemma 6.4, (48) and (49),

$$(80) \quad \Lambda(u) \leq \sum_{k < k_1 \in E_{x,y}^-(u)} \mathbb{P} \left(\sup_{(s,t) \in \widehat{I}_{k,0}(u)} Y(s,t) > u, \sup_{(s,t) \in \widehat{I}_{k_1,0}(u)} Y(s,t) > u \right) = o(\pi_5^+(u)), \quad u \rightarrow \infty, S \rightarrow \infty.$$

Moreover, it follows from Lemma 6.4 and (46) that

$$(81) \quad \pi_8(u) \leq \sum_{|k|=0}^{N_1(u)+2} \sum_{|l|=1, l \neq -1}^{N_2(u)+2} \mathbb{P} \left(\sup_{(s,t) \in I_{k,l}(u)} Y(s,t) > u \right) = o(\pi_5^+(u)), \quad u \rightarrow \infty, S \rightarrow \infty.$$

Inserting (76)–(81) into (74), we have

$$\pi_1(u) \geq 2\mathcal{H}_{\alpha_1} \frac{\overleftarrow{\rho}_2(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})} \Psi(u) \int_x^y \widehat{\mathcal{P}}_\beta^{g_t} e^{-\frac{\gamma_2 M_\beta}{\theta} t^\beta} dt (1+o(1)), \quad u \rightarrow \infty, S \rightarrow \infty,$$

and

$$\begin{aligned} \pi_1(u) &\leq 2\mathcal{H}_{\alpha_1} \frac{\overleftarrow{\rho}_2(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})} \Psi(u) \left(\int_x^y \widehat{\mathcal{P}}_\beta^{g_t} e^{-\frac{\gamma_2 M_\beta}{\theta} t^\beta} dt + \widehat{\mathcal{P}}_\beta^{\gamma_2 \kappa_1/2} \int_0^x e^{-\frac{\kappa_1}{2} \frac{\gamma_2}{\theta} t^\beta} dt \right. \\ (82) \quad &\quad \left. + \widehat{\mathcal{P}}_\beta^{\gamma_2 \kappa_1/2} \int_y^\infty e^{-\frac{\kappa_1}{2} \frac{\gamma_2}{\theta} t^\beta} dt \right) (1+o(1)), \end{aligned}$$

as $u \rightarrow \infty, S \rightarrow \infty$. Letting $x \rightarrow 0$ and $y \rightarrow \infty$ leads to

$$\pi_1(u) \sim 2\mathcal{H}_{\alpha_1} \frac{\overleftarrow{\rho}_2(u^{-1})}{\overleftarrow{\rho}_1(u^{-1})} \Psi(u) \int_0^\infty \widehat{\mathcal{P}}_\beta^{g_t} e^{-\frac{\gamma_2 M_\beta}{\theta} t^\beta} dt (1+o(1)), \quad u \rightarrow \infty, S \rightarrow \infty.$$

which, together with the fact that $\overleftarrow{\rho}_2(u^{-1}) \sim (\frac{\gamma_2}{\theta})^{1/\beta} \overleftarrow{v}_1(u^{-1})$, derives the claim. This completes the proof.

Case iii) $\gamma_1 = 0, \gamma_2 = \infty$ Let $X_z^\epsilon(s, t), (s, t) \in \mathbb{R}^2, z > 0, \epsilon > 0$ be homogeneous Gaussian random fields with correlation function

$$1 - \text{Corr}(X_z^\epsilon(s, t), X_z^\epsilon(s_1, t_1)) \sim (1 + \epsilon) \rho_1^2(|s - s_1|)(1 + o(1)) + \frac{1}{z} v_2^2(|t - t_1|)(1 + o(1)), \quad |s - s_1|, |t - t_1| \rightarrow \infty.$$

Thus, by Slepian inequality

$$(83) \quad \pi_9^+(u) \leq \pi_1(u) \leq \pi_9^-(u),$$

where

$$\begin{aligned}\pi_9^+(u) &:= \mathbb{P} \left(\sup_{|s| \leq \bar{v}_1(u^{-1})} X(s, \mu s) > u \right), \\ \pi_9^-(u) &:= \mathbb{P} \left(\sup_{(s,t) \in D_u} \frac{X_z^\epsilon(s,t)}{1 + (1-\epsilon)(v_1^2(|s+b_{12}t|) + v_2^2(|t|))} > u \right).\end{aligned}$$

It is straightforward to check that $\frac{X_z^\epsilon(s,t)}{1+(1-\epsilon)(v_1^2(|s+b_{12}t|)+v_2^2(|t|))}$ satisfies assumptions of Case ii) $\gamma_1 = 0, \gamma_2 = (1-\epsilon)z \in (0, \infty)$. Thus

$$\pi_9^-(u) \leq 2 \left(\frac{z}{\theta} \right)^{1/\beta} \mathcal{H}_{\alpha_1} \frac{\bar{v}_1(u^{-1})}{\bar{\rho}_1(u^{-1})} \Psi(u) \left(\int_0^\infty \hat{\mathcal{P}}_\beta^{g_{t,z}} e^{-\frac{zM_\beta}{\theta} t^\beta} dt \right) (1+o(1)), \quad u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0,$$

with

$$g_{s,z}(t) = \frac{z}{\theta} (|s+bt|^\beta + \theta|t|^\beta - |(1+b\mu)s|^\beta - \theta|\mu s|^\beta), \quad s \geq 0, t \in \mathbb{R}.$$

Replacing t by $z^{-1/\beta}s$ in the above integral yields

$$z^{1/\beta} \int_0^\infty \hat{\mathcal{P}}_\beta^{g_{t,z}} e^{-\frac{zM_\beta}{\theta} t^\beta} dt = \int_0^\infty \hat{\mathcal{P}}_\beta^{g_{z^{-1/\beta}s,z}} e^{-\frac{M_\beta}{\theta} s^\beta} ds.$$

Note that for any $\epsilon > 0$, there exists a positive constant $M_\epsilon > 0$ such that for z sufficiently large

$$g_{z^{-1/\beta}s,z}(t) + \epsilon|s|^\beta = \frac{1}{\theta} (|s+b_{12}tz^{1/\beta}|^\beta + \theta|tz^{1/\beta}|^\beta - |(1+b_{12}\mu)s|^\beta - \theta|\mu s|^\beta) + \epsilon|s|^\beta \geq M_\epsilon z|t|^\beta, \quad t \in \mathbb{R},$$

which implies that

$$\hat{\mathcal{P}}_\beta^{g_{z^{-1/\beta}s,z}} \leq e^{\epsilon|s|^\beta} \hat{\mathcal{P}}_\beta^{M_\epsilon z}.$$

Since

$$\lim_{z \rightarrow \infty} \hat{\mathcal{P}}_\beta^{M_\epsilon z} = 1,$$

then using dominated convergence theorem

$$\begin{aligned}\limsup_{z \rightarrow \infty} \int_0^\infty \hat{\mathcal{P}}_\beta^{g_{z^{-1/\beta}s,z}} e^{-\frac{M_\beta}{\theta} s^\beta} ds &\leq \limsup_{z \rightarrow \infty} \int_0^\infty \hat{\mathcal{P}}_\beta^{M_\epsilon z} e^{-\left(\frac{M_\beta}{\theta} - \epsilon\right)s^\beta} ds \\ &\rightarrow \left(\frac{M_\beta}{\theta} \right)^{-1/\beta} \Gamma(1/\beta + 1), \quad \epsilon \rightarrow 0.\end{aligned}$$

Thus we conclude that

$$(84) \quad \pi_9^-(u) \leq 2\Gamma(1/\beta + 1) (M_\beta)^{-1/\beta} \mathcal{H}_{\alpha_1} \frac{\bar{v}_1(u^{-1})}{\bar{\rho}_1(u^{-1})} \Psi(u) (1+o(1)), \quad u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0.$$

Next we focus on $\pi_9^+(u)$. One can easily check that the variance and correlation functions of $X(s, \mu s)$ satisfy

$$1 - \text{Var}(X(s, \mu s)) \sim v_1^2(|(1+b_{12}\mu)s|) + v_2^2(|\mu s|) \sim M_\beta v_1^2(|s|), \quad s \rightarrow 0,$$

and

$$1 - \text{Corr}(X(s, \mu s), X(s_1, \mu s_1)) \sim \rho_1^2(|s-s_1|) + \rho_2^2(|\mu(s-s_1)|) \sim \rho_1^2(|s-s_1|), \quad s, s_1 \rightarrow 0.$$

In light of Theorem 2.1, we have

$$\pi_9^+(u) \sim 2\Gamma(1/\beta + 1) (M_\beta)^{-1/\beta} \mathcal{H}_{\alpha_1} \frac{\bar{v}_1(u^{-1})}{\bar{\rho}_1(u^{-1})} \Psi(u), \quad u \rightarrow \infty,$$

which combined with (83) and (84) establishes the proof.

Case iv) $\gamma_1 \in (0, \infty), \gamma_2 = \infty$. Let $Z(s, t)$ be a homogeneous Gaussian random field with variance 1 and correlation function satisfying

$$1 - \text{Corr}(Z(s, t), Z(s_1, t_1)) \sim 2\rho_1^2(|s-s_1|) + \rho_1^2(|t-t_1|), \quad |s-s_1| \rightarrow 0, |t-t_1| \rightarrow 0,$$

$$\text{and } \hat{I}_{0,0}(u) = [-\bar{\rho}_1(u^{-1})S, \bar{\rho}_1(u^{-1})S] \times [-\bar{\rho}_1(u^{-1})S_1, \bar{\rho}_1(u^{-1})S_1].$$

Then, by Slepian's inequality and Lemma 6.4,

$$(85) \quad \pi_{10}^+(u) \leq \pi_1(u) \leq \pi_{10}^-(u) + \pi_{11}(u),$$

where

$$\begin{aligned}\pi_{10}^{\pm}(u) &= \mathbb{P} \left(\sup_{(s,t) \in \widehat{I}_{0,0}(u)} \frac{\overline{X}(s,t)}{1 + (1 \pm \epsilon) (v_1^2(|s + b_{12}t|) + v_2^2(|t|))} > u \right), \\ \pi_{11}(u) &= \mathbb{P} \left(\sup_{(s,t) \in (D_u - \widehat{I}_{0,0}(u))} \frac{Z(s,t)}{1 + \frac{\kappa_1}{2} (v_1^2(|s|) + v_2^2(|t|))} > u \right).\end{aligned}$$

Note that $\rho_2^2(t) = o(\rho_1^2(t))$ as $t \rightarrow 0$ and

$$(1 \pm \epsilon)u^2 (v_1^2(|\widehat{p}_1(u^{-1})s + b_{12}\widehat{p}_1(u^{-1})t|) + v_2^2(|\widehat{p}_1(u^{-1})t|)) \rightarrow (1 \pm \epsilon)\gamma_1 (|s + b_{12}t|^{\alpha_1} + \theta|t|^{\alpha_1}), \quad u \rightarrow \infty$$

uniformly with respect to $(s,t) \in [-S, S] \times [-S_1, S_1]$. It follows from Lemma 5.3 that

$$\pi_{10}^{\pm}(u) \sim \Psi(u) \mathbb{E} \left\{ \exp \sup_{(s,t) \in [-S, S] \times [-S_1, S_1]} \left[\sqrt{2}B_{\alpha_1}(s) - |s|^{\alpha_1} - (1 \pm \epsilon)\gamma_1 (|s + b_{12}t|^{\alpha_1} + \theta|t|^{\alpha_1}) \right] \right\}.$$

Since

$$\lim_{S_1 \rightarrow \infty} \mathbb{E} \left\{ \exp \sup_{(s,t) \in [-S, S] \times [-S_1, S_1]} \left[\sqrt{2}B_{\alpha_1}(s) - |s|^{\alpha_1} - (1 \pm \epsilon)\gamma_1 (|s + b_{12}t|^{\alpha_1} + \theta|t|^{\alpha_1}) \right] \right\} = \widehat{\mathcal{P}}_{\alpha_1}^{(1 \pm \epsilon)\gamma_1 M_{\alpha_1}}(S),$$

we have

$$\pi_{10}^{\pm}(u) \sim \widehat{\mathcal{P}}_{\alpha_1}^{\gamma_1 M_{\alpha_1}} \Psi(u), \quad u \rightarrow \infty, S_1 \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0.$$

Using that $\frac{Z(s,t)}{1 + \frac{\kappa_1}{2} (v_1^2(|s|) + v_2^2(|t|))}$ satisfies the conditions of Case iii) $\gamma_1, \gamma_2 \in (0, \infty)$ of Theorem 3.1, by the same argument as given in the proof of (53), we obtain that $\pi_{11}(u) = o(\Psi(u))$, $u \rightarrow \infty, S_1 \rightarrow \infty, S \rightarrow \infty$. Thus the proof is completed.

Case iii) $\gamma_1 = \gamma_2 = \infty$. It follows from (56) and (59) with the specific B in this case that

$$\mathbb{P}(X(0,0) > u) \leq \pi_1(u) \leq \mathbb{P} \left(\sup_{(s,t) \in D_u} Y(s,t) > u \right),$$

where κ_1 is defined in Lemma 6.4. The Gaussian random field involved in the right hand side of the above inequality satisfies the assumption of Case iii) $\gamma_1 = \gamma_2 = \infty$ in Theorem 3.1 and therefore it follows that

$$\mathbb{P} \left(\sup_{(s,t) \in D_u} \frac{\overline{X}(s,t)}{1 + \frac{\kappa_1}{2} (v_1^2(|s|) + v_2^2(|t|))} > u \right) \sim \Psi(u), \quad u \rightarrow \infty.$$

This completes the proof. \square

Proof of Theorem 3.8 For $\epsilon > 0$ sufficiently small, let $Z^{\pm\epsilon}$ be a stationary Gaussian process with continuous trajectories, unit variance and correlation function satisfying

$$1 - r_{Z^{\pm\epsilon}}(t) \sim (1 \mp \epsilon)\rho_1^2(|t|), \quad t \rightarrow 0.$$

By Slepian's inequality, we have

$$\pi_{12}^+(u) \leq \pi_1(u) \leq \pi_{12}^-(u),$$

where

$$\pi_{12}^{\pm}(u) = \mathbb{P} \left(\sup_{(s,t) \in D_u} \frac{Z^{\pm\epsilon}(s)}{1 + (1 \pm \epsilon) (v_1^2(|s|) + v_2^2(|t|))} > u \right).$$

By the fact that for any $u > 0$,

$$\sup_{(s,t) \in D_u} \frac{Z^{\pm\epsilon}(s)}{1 + (1 \pm \epsilon) (v_1^2(|s|) + v_2^2(|t|))} = \sup_{|s| \leq \widehat{b}_1(u^{-1} \ln u)} \frac{Z^{\pm\epsilon}(s)}{1 + (1 \pm \epsilon) v_1^2(|s|)},$$

we have

$$\pi_{12}^{\pm}(u) = \mathbb{P} \left(\sup_{|s| \leq \widehat{b}_1(u^{-1} \ln u)} \frac{Z^{\pm\epsilon}(s)}{1 + (1 \pm \epsilon) v_1^2(|s|)} > u \right).$$

Applying Theorem 2.1, we establish the claims. \square

Proof of Theorem 3.9 Set below for $u > 0$

$$D_u = \{|s| \leq \check{v}_1(u^{-1} \ln u), |t| \leq 2\mu \check{v}_1(u^{-1} \ln u)\}.$$

Using the same $Z^{\pm\epsilon}$ as in the proof of Theorem 3.8, by Slepian's inequality, we have

$$\pi_{13}^+(u) \leq \pi_1(u) \leq \pi_{13}^-(u),$$

where

$$\pi_{13}^\pm(u) = \mathbb{P} \left(\sup_{(s,t) \in D_u} \frac{Z^{\pm\epsilon}(s)}{1 + (1 \pm \epsilon)(v_1^2(|s + b_{12}t|) + v_2^2(|t|))} > u \right).$$

The same analysis as given between (92) and (93) implies that, for u sufficiently large

$$(1 - \epsilon)M_{\beta_1}v_1^2(|s|) \leq \inf_{|t| \leq 2\mu \check{v}_1(u^{-1} \ln u)} v_1^2(|s + b_{12}t|) + v_2^2(|t|) \leq (1 + \epsilon)M_{\beta_1}v_1^2(|s|)$$

hold for $|s| \leq \check{v}_1(u^{-1} \ln u)$. Thus we have

$$\pi_{13}^-(u) \leq \mathbb{P} \left(\sup_{|s| \leq \check{v}_1(u^{-1} \ln u)} \frac{Z^{-\epsilon}(s)}{1 + (1 - \epsilon)^2 M_{\beta_1} v_1^2(|s|)} > u \right),$$

and

$$\pi_{13}^+(u) \geq \mathbb{P} \left(\sup_{|s| \leq \check{v}_1(u^{-1} \ln u)} \frac{Z^{+\epsilon}(s)}{1 + (1 + \epsilon)^2 M_{\beta_1} v_1^2(|s|)} > u \right).$$

Hence the claim follows by Theorem 2.1. \square

5. APPENDIX A

In this section we derive some key uniform expansions of the tail of maximum of Gaussian random fields over short intervals. For any $\gamma \in (0, \infty)$, $S > 0$ we define

$$\mathcal{P}_\alpha^\gamma = \mathbb{E} \left\{ \sup_{[0, S]} e^{\sqrt{2}B_\alpha(t) - (1+\gamma)|t|^\alpha} \right\}$$

and we set

$$\mathcal{P}_\alpha^\infty[0, S] = 1, \quad \mathcal{P}_\alpha^0[0, S] = \mathcal{H}_\alpha[0, S], \quad \alpha \in (0, 2], S > 0.$$

The claim of the following three lemmas follows by Theorem 2.1 in [12]; the detailed proofs are omitted here.

In the following $h_k, k \in \mathcal{K}_u$ with \mathcal{K}_u an index set are positive functions such that $\lim_{u \rightarrow \infty} h_k(u)/u = 1$ uniformly with respect to $k \in \mathcal{K}_u$.

Lemma 5.1. *Let $X_{u,k}(t), t \in [0, T], k \in \mathcal{K}_u$ be a sequence of centered Gaussian processes with continuous trajectories, variance 1 and correlation function $r(\cdot, \cdot)$ satisfying (8) uniformly with respect to $k \in \mathcal{K}_u$. Suppose that $\rho \in \mathcal{R}_\alpha/2, v \in \mathcal{R}_{\beta/2}$ with $0 < \alpha \leq 2, \beta > 0$. If $\lim_{t \rightarrow 0} \frac{v^2(t)}{\rho^2(t)} = \gamma \in [0, \infty]$, then*

$$\lim_{u \rightarrow \infty} \sup_{k \in \mathcal{K}_u} \left| \frac{1}{\Psi(h_k(u))} \mathbb{P} \left\{ \sup_{t \in [0, \check{p}(u^{-1})S]} \frac{X_{u,k}(t)}{1 + v^2(t)} > h_k(u) \right\} - \mathcal{P}_\alpha^\gamma[0, S] \right| = 0.$$

Let $\rho_i \in \mathcal{R}_{\alpha_i/2}, v_i \in \mathcal{R}_{\beta_i/2}, i = 1, 2$ be non-negative functions with $0 < \alpha_i \leq 2, \beta_i > 0, i = 1, 2$. Let $X_{u,k}(s, t), k \in \mathcal{K}_u$ be centered Gaussian random fields over $\mathcal{E}(u) := \{(\check{p}_1(u^{-1})s, \check{p}_2(u^{-1})t), (s, t) \in \mathcal{E}\}$ with \mathcal{E} an compact set containing 0. Suppose further that $X_{u,k}$ has unit variance, continuous trajectories and correlation function $r_k(s, t, s_1, t_1)$ satisfying (14) uniformly with respect to $k \in \mathcal{K}_u$.

Lemma 5.2. *Let $d_u(s, t), u > 0$ be continuous functions satisfying*

$$(86) \quad \lim_{u \rightarrow \infty} \sup_{(s, t) \in \mathcal{E}, k \in \mathbb{K}_u} |h_k^2(u) d_u(\overleftarrow{p}_1(u^{-1})s, \overleftarrow{p}_2(u^{-1})t) - d(s, t)| = 0.$$

If further $d_u(s, t) > -1$ for any $s, t \in \mathcal{E}$ and all u large, then

$$\lim_{u \rightarrow \infty} \sup_{k \in \mathbb{K}_u} \left| \frac{1}{\Psi(h_k(u))} \mathbb{P} \left(\sup_{(s, t) \in \mathcal{E}(u)} \frac{X_{u, k}(s, t)}{1 + d_u(s, t)} > h_k(u) \right) - \mathbb{E} \left\{ e^{\sup_{(s, t) \in \mathcal{E}} \{W_{\alpha_1, \alpha_2}(s, t) - d(s, t)\}} \right\} \right| = 0,$$

where $W_{\alpha_1, \alpha_2}(s, t) = \sqrt{2}(B_{\alpha_1}(s) + \tilde{B}_{\alpha_2}(t)) - |s|^{\alpha_1} - |t|^{\alpha_2}$, $s, t \in \mathbb{R}$, with B_{α_1} and B_{α_2} being two independent fBm's with indices α_1, α_2 , respectively.

Lemma 5.3. *Suppose that $d_u(s, t), u > 0$ are continuous functions satisfying*

$$\lim_{u \rightarrow \infty} \sup_{(s, t) \in \mathcal{E}, k \in \mathbb{K}_u} |h_k^2(u) d_u(\overleftarrow{p}_1(u^{-1})s, \overleftarrow{p}_1(u^{-1})t) - d(s, t)| = 0.$$

If $\rho_2^2(t) = o(\rho_1^2(t))$ as $t \rightarrow 0$ and $d_u(s, t) > -1$ for any $s, t \in \mathcal{E}$ and all u large, then

$$\lim_{u \rightarrow \infty} \sup_{k \in \mathbb{K}_u} \left| \frac{1}{\Psi(h_k(u))} \mathbb{P} \left(\sup_{(s, t) \in \tilde{\mathcal{E}}(u)} \frac{X_{u, k}(s, t)}{1 + d_u(s, t)} > h_k(u) \right) - \mathbb{E} \left\{ \sup_{(s, t) \in \mathcal{E}} e^{\sqrt{2}B_{\alpha_1}(s) - |s|^{\alpha_1} - d(s, t)} \right\} \right| = 0,$$

with B_{α_1} an fBm with index α_1 and $\tilde{\mathcal{E}}(u) := \{(\overleftarrow{p}_1(u^{-1})s, \overleftarrow{p}_1(u^{-1})t), (s, t) \in \mathcal{E}\}$.

Assume now that $X(t), t = (t_1, \dots, t_d) \in \mathbb{R}^d$ is a Gaussian field with continuous trajectories, unit variance and covariance function satisfying

$$(87) \quad 1 - \text{Cov}(X(s), X(t)) \sim \sum_{i=1}^d \rho_i(|t_i - s_i|), \quad s, t \rightarrow 0,$$

with ρ_i positive regularly varying function with index $\alpha_i/2 \in (0, 1]$. Denote by $\overleftarrow{p}_d(u^{-1}) = (\overleftarrow{p}_1(u^{-1}), \dots, \overleftarrow{p}_d(u^{-1}))$ and define $\overleftarrow{p}_d(u^{-1})t = (\overleftarrow{p}_1(u^{-1})t_1, \dots, \overleftarrow{p}_d(u^{-1})t_d)$. Moreover, set $F(A, B) = \inf_{s \in A, t \in B} \sum_{i=1}^d |s_i - t_i|$ for any $A, B \subset \mathbb{R}^d$ and let

$$D_u = \prod_{i=1}^d \left[-\frac{\delta_u}{\overleftarrow{p}_i(u^{-1})}, \frac{\delta_u}{\overleftarrow{p}_i(u^{-1})} \right], \quad \mathbb{K} = \{(\lambda_1, \lambda_2) \in D_u \times D_u, \lambda_1 + \mathcal{E}_1, \lambda_2 + \mathcal{E}_2 \subset D_u\}.$$

Further, let $u_\lambda, \lambda \in D_u$, with $\delta_u \rightarrow 0, u \rightarrow \infty$ satisfy

$$\lim_{u \rightarrow \infty} \sup_{\lambda \in D_u} \left| \frac{u_\lambda}{u} - 1 \right| = 0.$$

We state next the result of Corollary 3.2 in [12], below $\mathcal{E}_1, \mathcal{E}_2$ are assumed to be compact sets.

Lemma 5.4. *Suppose that $X(t), t = (t_1, \dots, t_d) \in \mathbb{R}^d$ is a Gaussian field with continuous trajectories, unit variance and covariance function satisfying (87). Then there exists $\mathcal{C}, \mathcal{C}_1 > 0$ such that for any $S > 1$ as u sufficiently large,*

$$\sup_{(\lambda_1, \lambda_2) \in \mathbb{K}, \mathcal{E}_1, \mathcal{E}_2 \subset [0, S]^d} e^{\frac{\mathcal{C}_1 F^{\beta^*}/2(\lambda_1 + \mathcal{E}_1, \lambda_2 + \mathcal{E}_2)}{16}} \frac{\mathbb{P} \left\{ \sup_{t \in \overleftarrow{p}_d(u^{-1})(\lambda_1 + \mathcal{E}_1)} X(t) > u_{\lambda_1}, \sup_{s \in \overleftarrow{p}_d(u^{-1})(\lambda_2 + \mathcal{E}_2)} X(s) > u_{\lambda_2} \right\}}{S^{2d} \Psi(u_{\lambda_1, \lambda_2}(u))} \leq \mathcal{C}$$

with $u_{\lambda_1, \lambda_2} = \min(u_{\lambda_1}, u_{\lambda_2})$ and $\beta^* = \min_{i=1, \dots, d} \alpha_i$.

6. APPENDIX B

Let in this section g be a positive function such that

$$\lim_{u \rightarrow \infty} g(u) = \infty, \quad \lim_{u \rightarrow \infty} \frac{g(u)}{u} = 0.$$

Further, let $v \in \mathcal{R}_\beta, \beta > 0$ be a non-negative function. We set throughout in the following

$$(88) \quad t(u) := \overleftarrow{v} \left(\frac{g(u)}{u} \right), \quad u > 0.$$

We shall investigate first the asymptotic behaviour of an integral determined by g and v .

Lemma 6.1. *i) For any $0 < x \leq y < \infty$ and $c > 0$, as $u \rightarrow \infty$*

$$\int_0^{xt(u)} e^{-cu^2v^2(t)} dt \sim \int_0^{yt(u)} e^{-cu^2v^2(t)} dt.$$

ii) If $a \in \mathcal{R}_\beta$ is such that $a(t) \sim v(t)$ as $t \rightarrow 0$, then as $u \rightarrow \infty$

$$\int_0^{t(u)} e^{-u^2v^2(t)} dt \sim \int_0^{\overleftarrow{a}(u^{-1}g(u))} e^{-u^2a^2(t)} dt.$$

Proof of Lemma 6.1 i) Using standard properties of regularly varying functions, see e.g., [19], for u sufficiently large and $0 < x < y < \infty$, we have

$$\begin{aligned} \int_{xt(u)}^{yt(u)} e^{-cu^2v^2(t)} dt &\leq e^{-cu^2v^2((x/2)t(u))} (y-x)t(u) \\ &\leq e^{-(x/3)^{2\beta}cu^2v^2(t(u))} (y-x)t(u) \\ &\leq e^{-(x/4)^{2\beta}c(g(u))^2} (y-x)t(u) \end{aligned}$$

and

$$\begin{aligned} \int_0^{xt(u)} e^{-cu^2v^2(t)} dt &\geq \int_0^{(x/8)t(u)} e^{-cu^2v^2(t)} dt \\ &\geq e^{-cu^2v^2((x/7)t(u))} (x/8)t(u) \\ &\geq e^{-(x/6)^{2\beta}cu^2v^2(t(u))} (x/8)t(u) \\ &\geq e^{-(x/5)^{2\beta}c(g(u))^2} (x/8)t(u), \end{aligned}$$

which imply that, as $u \rightarrow \infty$,

$$\int_0^{xt(u)} e^{-cu^2v^2(t)} dt \sim \int_0^{yt(u)} e^{-cu^2v^2(t)} dt.$$

ii) For any $0 < \epsilon < 1/2$

$$(1-\epsilon)a(t) \leq v(t) \leq (1+\epsilon)a(t)$$

holds for t sufficiently small. Consequently, for u sufficiently large

$$\begin{aligned} \int_0^{t(u)} e^{-u^2v^2(t)} dt &\leq \int_0^{t(u)} e^{-(1-\epsilon)^2u^2a^2(t)} dt \leq \int_0^{t(u)} e^{-u^2a^2((1-2\epsilon)^{1/\beta}t)} dt \\ &= (1-2\epsilon)^{-1/\beta} \int_0^{(1-2\epsilon)^{1/\beta}t(u)} e^{-u^2a^2(t)} dt \\ &\leq (1-2\epsilon)^{-1/\beta} \int_0^{t(u)} e^{-u^2a^2(t)} dt \end{aligned}$$

and

$$\begin{aligned} \int_0^{t(u)} e^{-u^2v^2(t)} dt &\geq \int_0^{t(u)} e^{-(1+\epsilon)^2u^2a^2(t)} dt \geq \int_0^{t(u)} e^{-u^2a^2((1+2\epsilon)^{1/\beta}t)} dt \\ &= (1+2\epsilon)^{-1/\beta} \int_0^{(1+2\epsilon)^{1/\beta}t(u)} e^{-u^2a^2(t)} dt \\ &\geq (1+2\epsilon)^{-1/\beta} \int_0^{t(u)} e^{-u^2a^2(t)} dt. \end{aligned}$$

Letting $\epsilon \rightarrow 0$ and by the fact that $\overleftarrow{a}(u^{-1}g(u)) \sim t_u$, we establish the second claim. \square

Lemma 6.2. *We have*

$$(89) \quad \int_0^{t(u)} e^{-cu^2v^2(t)} dt \sim c^{-1/(2\beta)} \Gamma(1 + 1/(2\beta)) \overleftarrow{v}(1/u), \quad u \rightarrow \infty.$$

Proof of Lemma 6.2 By Lemma 6.1, ii) we can assume that $v(x) = \ell(x)x^\beta$ with ℓ normalized slowly varying function at 0. It is well-known that $\ell(x)x^\beta$ is ultimately monotone for any $\beta \neq 0$, ℓ is continuously differentiable and

$$(90) \quad \lim_{x \rightarrow 0} \frac{x\ell'(x)}{\ell(x)} = 0.$$

Since v is ultimately monotone, we have with $g(u)$ and $t(u)$ defined by (88)

$$(91) \quad \int_0^{t(u)} e^{-cu^2v^2(t)} dt \sim u^{-1} \int_0^{g(u)} \frac{1}{v'(\overleftarrow{v}(y/u))} e^{-cy^2} dy, \quad u \rightarrow \infty.$$

Further, (90) implies

$$\frac{1}{v'(\overleftarrow{v}(y/u))} \sim \frac{1}{\beta} \frac{\overleftarrow{v}(y/u)}{v(\overleftarrow{v}(y/u))} \sim \frac{1}{\beta} \frac{u}{y} \overleftarrow{v}(y/u)$$

Consequently, as $u \rightarrow \infty$

$$\begin{aligned} \int_0^{t(u)} e^{-cu^2v^2(t)} dt &\sim \frac{1}{\beta} \int_0^{g(u)} \overleftarrow{v}(y/u) y^{-1} e^{-cy^2} dy \\ &\sim \frac{1}{\beta} \overleftarrow{v}(1/u) \int_0^{g(u)} \frac{\overleftarrow{v}(y/u)}{\overleftarrow{v}(1/u)} y^{-1} e^{-cy^2} dy. \end{aligned}$$

Potter's theorem shows that there exists a constant C such that for u sufficiently large,

$$\frac{\overleftarrow{v}(y/u)}{\overleftarrow{v}(1/u)} \leq C(\max(1, y))^{2/\beta}, \quad 0 \leq y \leq g(u).$$

By the fact that for any $y > 0$

$$\lim_{u \rightarrow \infty} \frac{\overleftarrow{v}(y/u)}{\overleftarrow{v}(1/u)} = y^{1/\beta}$$

and the dominated convergence theorem, since $\lim_{u \rightarrow \infty} g(u) = \infty$, we obtain

$$\begin{aligned} \int_0^{t(u)} e^{-u^2v^2(t)} dt &\sim \frac{1}{\beta} \overleftarrow{v}(1/u) \int_0^\infty y^{1/\beta-1} e^{-cy^2} dy \\ &\sim c^{-1/(2\beta)} \Gamma(1 + 1/(2\beta)) \overleftarrow{v}(1/u). \end{aligned}$$

Note that alternatively, by [36][Proposition 1.18] it follows that

$$\int_0^{g(u)} \frac{\overleftarrow{v}(y/u)}{\overleftarrow{v}(1/u)} y^{-1} e^{-cy^2} dy \sim \int_0^{g(u)} y^{1/\beta-1} e^{-cy^2} dy, \quad u \rightarrow \infty$$

and thus again the claim follows. \square

Lemma 6.3. Suppose that $\rho_1^2 \in \mathcal{R}_{\alpha_1}$ and $\rho_2^2 \in \mathcal{R}_{\alpha_2}$ with $\alpha_1, \alpha_2 > 0$. If $\rho_1^2(|t|) = o(\rho_2^2(|t|))$ as $t \rightarrow 0$, then for any $a, b \in \mathbb{R}$,

$$\rho_1^2(|as + bt|) + \rho_2^2(|t|) \sim \rho_1^2(|as|) + \rho_2^2(|t|), \quad s, t \rightarrow 0.$$

Proof of Lemma 6.3 The claim follows easily if $abst = 0$. Next we suppose that $abst \neq 0$. It suffices to prove that

$$\lim_{s, t \rightarrow 0, st \neq 0} \frac{|\rho_1^2(|as + bt|) - \rho_1^2(|as|)|}{\rho_1^2(|as|) + \rho_2^2(|t|)} = 0.$$

For any $\epsilon \in (0, \alpha_1)$, if $|\frac{as}{bt}| > \frac{4\alpha_1}{\epsilon}$, then

$$1 - \frac{\epsilon}{4\alpha_1} \leq \frac{|as + bt|}{|as|} \leq 1 + \frac{\epsilon}{4\alpha_1}.$$

Thus in light of UCT, we have, for s, t sufficiently small

$$\begin{aligned} \frac{|\rho_1^2(|as + bt|) - \rho_1^2(|as|)|}{\rho_1^2(|as|) + \rho_2^2(|t|)} &\leq \frac{\rho_1^2(|as|) \left| \frac{\rho_1^2(|as+bt|)}{\rho_1^2(|as|)} - 1 \right|}{\rho_1^2(|as|) + \rho_2^2(|t|)} \leq \left| \frac{\rho_1^2(|as + bt|)}{\rho_1^2(|as|)} - 1 \right| \\ &\leq \max \left(\left(1 + \frac{\epsilon}{2\alpha_1} \right)^{\alpha_1} - 1, 1 - \left(1 - \frac{\epsilon}{2\alpha_1} \right)^{\alpha_1} \right) =: b_\epsilon. \end{aligned}$$

For any $\epsilon \in (0, \alpha_1)$, if $|\frac{as}{bt}| \leq \frac{4\alpha_1}{\epsilon}$, then

$$\frac{|as + bt|}{|bt|} \leq 1 + \frac{4\alpha_1}{\epsilon}.$$

Applying again UCT we obtain

$$\begin{aligned} \frac{|\rho_1^2(as + bt) - \rho_1^2(as)|}{\rho_1^2(as) + \rho_2^2(|t|)} &\leq \frac{\rho_1^2(|bt|)}{\rho_1^2(as) + \rho_2^2(|t|)} \left| \frac{\rho_1^2(as + bt)}{\rho_1^2(|bt|)} - \frac{\rho_1^2(as)}{\rho_1^2(|bt|)} \right| \\ &\leq \frac{\rho_1^2(|bt|)}{\rho_2^2(|t|)} \left[\left(1 + \frac{8\alpha_1}{\epsilon}\right)^{\alpha_1} + \left(\frac{8\alpha_1}{\epsilon}\right)^{\alpha_1} \right] \\ &\rightarrow 0, \text{ as } st \neq 0, s, t \rightarrow 0. \end{aligned}$$

Consequently,

$$\lim_{s, t \rightarrow 0, st \neq 0} \frac{|\rho_1^2(as + bt) - \rho_1^2(as)|}{\rho_1^2(as) + \rho_2^2(|t|)} \leq b_\epsilon \rightarrow 0, \quad \epsilon \rightarrow 0.$$

This completes the proof. \square

Lemma 6.4. Suppose that $v_1^2, v_2^2 \in \mathcal{R}_\beta$ $\beta > 0$. If $a_1 v_2^2(|t|) \leq v_1^2(|t|) \leq a_2 v_2^2(|t|)$ with $a_1, a_2 > 0$ for t sufficiently small, then for any reversible matrix $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, there exist two positive constants κ_1 and κ_2 such that

$$\kappa_1 v_1^2(|s|) + \kappa_1 v_2^2(|t|) \leq v_1^2(|b_{11}s + b_{12}t|) + v_2^2(|b_{21}s + b_{22}t|) \leq \kappa_2 v_1^2(|s|) + \kappa_2 v_2^2(|t|)$$

holds in a neighbourhood of 0.

Proof of Lemma 6.4 Without loss of generality, we assume that $|t| \geq |s|$ and $|t| > 0$. By UCT, we have

$$\begin{aligned} \frac{v_1^2(|b_{11}s + b_{12}t|) + v_2^2(|b_{21}s + b_{22}t|)}{v_1^2(|s|) + v_2^2(|t|)} &\leq \frac{a_2 v_2^2(|t|(|b_{12}| + |b_{11}\frac{s}{t}|)) + v_2^2(|t|(|b_{22}| + |b_{21}\frac{s}{t}|))}{v_2^2(|t|)} \\ &\leq 2(a_2(|b_{11}| + |b_{12}|)^\beta + (|b_{21}| + |b_{22}|)^\beta), \end{aligned}$$

for t sufficiently small. Hence we get the upper bound. For the lower bound, making a linear transformation

$$(s, t)^\top = B^{-1}(s', t')^\top = \begin{pmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \end{pmatrix} (s', t')^\top,$$

and then using the above conclusion, we have

$$\begin{aligned} v_1^2(|s|) + v_2^2(|t|) &= v_1^2(|b'_{11}s' + b'_{12}t'|) + v_2^2(|b'_{21}s' + b'_{22}t'|) \\ &\leq 2(a_2(|b'_{11}| + |b'_{12}|)^\beta + (|b'_{21}| + |b'_{22}|)^\beta) (v_1^2(|s'|) + v_2^2(|t'|)) \\ &\leq 2(a_2(|b'_{11}| + |b'_{12}|)^\beta + (|b'_{21}| + |b'_{22}|)^\beta) (v_1^2(|b_{11}s + b_{12}t|) + v_2^2(|b_{21}s + b_{22}t|)), \end{aligned}$$

provided $|t'| \geq |s'|$ and $|t'| > 0$ for t' sufficiently small. This completes the proof. \square

Proof of (75). Note that

$$(92) \quad v_1^2(|s + b_{12}t|) + v_2^2(|t|) = v_1^2(|s|) \frac{v_1^2(|s||1 + b_{12}t/s|) + v_2^2(|s||t/s|)}{v_1^2(|s|)}, \quad (s, t) \in D_u.$$

If $|t/s| \leq M < \infty$, then by UCT

$$\sup_{(s, t) \in D_u, |t/s| \leq M} \left| \frac{v_1^2(|s||1 + b_{12}t/s|) + v_2^2(|s||t/s|)}{v_1^2(|s|)} - |1 + b_{12}t/s|^\beta - \theta|t/s|^\beta \right| \rightarrow 0, \quad u \rightarrow \infty.$$

If $|t/s| \geq M$, then using Potter's bound, for u sufficiently large

$$\inf_{(s, t) \in D_u, |t/s| \geq M} \frac{v_1^2(|s||1 + b_{12}t/s|) + v_2^2(|s||t/s|)}{v_1^2(|s|)} \geq 1/2 \left(|b_{12}M - 1|^{\beta/2} + \theta M^{\beta/2} \right).$$

Therefore, the minimum of $v_1^2(|s+b_{12}t|)+v_2^2(|t|)$ is attained for $|t/s| \leq M$ with M sufficiently large. Further, the minimum of $|1+b_{12}t/s|^\beta + \theta|t/s|^\beta$ is attained at $\mu := t/s \in [-1/|b_{12}|, 1/|b_{12}|]$. Thus, for $(s, t) \in D_u$ and u sufficiently large

$$(93) \quad \begin{aligned} \frac{v_1^2(|s+b_{12}t|)+v_2^2(|t|)}{v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)} &= \frac{v_1^2(|s+b_{12}t|)+v_2^2(|t|)}{v_1^2(|s|)} \frac{v_1^2(|s|)}{v_1^2(|(1+b_{12}\mu)s|)+v_2^2(|\mu s|)} \\ &\geq \frac{|1+b_{12}\mu|^\beta + \theta|\mu|^\beta}{2} \frac{1}{2(|1+b_{12}\mu|^\beta + \theta|\mu|^\beta)} = 1/4, \end{aligned}$$

Recall that $v(s, t) = v_1^2(|s+b_{12}t|) + v_2^2(|t|) - v_1^2(|(1+b_{12}\mu)s|) - v_2^2(|\mu s|)$, $(s, t) \in D_u$. Note that $v(s, t)$ may be negative at some point. It follows that

$$\begin{aligned} &[1 + (1 - 2\epsilon)(v_1^2(|(1+b_{12}\mu)s|) + v_2^2(|\mu s|))] [1 + (1 - 2\epsilon)v(s, t)] \\ &= 1 + (1 - 2\epsilon)(v_1^2(|s+b_{12}t|) + v_2^2(|t|)) + (1 - 2\epsilon)^2(v_1^2(|(1+b_{12}\mu)s|) + v_2^2(|\mu s|))v(s, t). \end{aligned}$$

Moreover (93) yields that

$$(v_1^2(|(1+b_{12}\mu)s|) + v_2^2(|\mu s|))v(s, t) = o(v_1^2(|s+b_{12}t|) + v_2^2(|t|)), \quad (s, t) \in D_u, u \rightarrow \infty.$$

Thus, we have for any $0 < \epsilon < 1/4$ and sufficiently large u

$$(94) \quad [1 + (1 - 2\epsilon)(v_1^2(|(1+b_{12}\mu)s|) + v_2^2(|\mu s|))] [1 + (1 - 2\epsilon)v(s, t)] \leq 1 + (1 - \epsilon)(v_1^2(|s+b_{12}t|) + v_2^2(|t|)),$$

with $(s, t) \in D_u$. Since for $|s| \in [\check{\rho}_2(u^{-1})x/2, \check{\rho}_2(u^{-1})2y]$, $|t| \in [M\check{\rho}_2(u^{-1}), \check{\rho}_2(\ln u/u)]$

$$\begin{aligned} v(s, t) &= v_1^2(|t|) \frac{v_1^2(|s+b_{12}t|) + v_2^2(|t|) - v_1^2(|(1+b_{12}\mu)s|) - v_2^2(|\mu s|)}{v_1^2(|t|)} \\ &\sim v_1^2(|t|) \left(\left| b_{12} + \frac{s}{t} \right|^\beta + \theta - |1+b_{12}\mu|^\beta \left| \frac{s}{t} \right|^\beta - \theta \left| \mu \frac{s}{t} \right|^\beta \right), \quad u \rightarrow \infty \end{aligned}$$

then for M, u sufficiently large

$$(95) \quad v(s, t) \geq \frac{1-3\epsilon}{1-2\epsilon}v(s_1, t_1),$$

with $|s|, |s_1| \in [\check{\rho}_2(u^{-1})x/2, \check{\rho}_2(u^{-1})2y]$, $|t|, |t_1| \in [M\check{\rho}_2(u^{-1}), \check{\rho}_2(\ln u/u)]$.

Moreover, for any $\epsilon_1 > 0$, $|s| \in [\check{\rho}_2(u^{-1})x/2, \check{\rho}_2(u^{-1})2y]$ and $|t| \in [0, M\check{\rho}_2(u^{-1})]$, by UCT

$$\begin{aligned} v(s, t) &\geq v_1^2(|s|) \left[(1 - \epsilon_1) \left(|1+b_{12}t/s|^\beta + \theta|t/s|^\beta \right) - (1 + \epsilon_1) (|1+b_{12}\mu|^\beta + \theta|\mu|^\beta) \right] \\ &\geq -2\epsilon_1 (|1+b_{12}\mu|^\beta + \theta|\mu|^\beta) v_1^2(|s|), \end{aligned}$$

and for any $|s|, |s_1| \in [\frac{i-1}{n}\check{\rho}_2(u^{-1}), \frac{i+2}{n}\check{\rho}_2(u^{-1})]$ with $x/2 \leq \frac{i}{n} \leq 2y$ and $|t| \in [0, M\check{\rho}_2(u^{-1})]$ and u and n sufficiently large

$$\begin{aligned} |v(s, t) - v(s_1, t)| &\leq v_1^2(|s|) \sup_{d_1, d_2 \in \{\pm \epsilon_1\}} |(1+d_1)(|1+b_{12}t/s|^\beta + |1+b_{12}\mu|s_1/s|^\beta + \theta|\mu s_1/s|^\beta) \\ &\quad - (1+d_2)(|1+b_{12}\mu|^\beta + \theta|\mu|^\beta + |s_1/s+b_{12}t/s|^\beta)| \\ &\leq v_1^2(|s|) \mathbb{Q}\epsilon_1 (|1+b_{12}\mu|^\beta + \theta|\mu|^\beta + |1+2|b_{12}|M/x|^\beta) + v_1^2(|s|)|s_1/s|^\beta - 1 (|1+b_{12}\mu|^\beta + \theta|\mu|^\beta) \\ &\quad + v_1^2(|s|) \sup_{|z| \in [0, 4M/x]} |h_{s_1/s}(z) - h_1(z)|, \end{aligned}$$

where $h_s(z) = |s+b_{12}z|^\beta$, $s, z \in \mathbb{R}$. Therefore, for $|s|, |s_1| \in [\frac{i-1}{n}\check{\rho}_2(u^{-1}), \frac{i+2}{n}\check{\rho}_2(u^{-1})]$ with $x/2 \leq \frac{i}{n} \leq 2y$ and $|t| \in [0, M\check{\rho}_2(u^{-1})]$ when ϵ_1 sufficiently small and u and n sufficiently large

$$v(s, t) \geq -\epsilon/4 (|1+b_{12}\mu|^\beta + \theta|\mu|^\beta) v_1^2(|s|), \quad |v(s, t) - v(s_1, t)| \leq \epsilon/8 (|1+b_{12}\mu|^\beta + \theta|\mu|^\beta) v_1^2(|s|),$$

which implies that (recall that $\lim_{u \rightarrow \infty} \sup_{(s, t) \in D_u} |v(s, t)| = 0$)

$$\begin{aligned} &\epsilon(v_1^2(|(1+b_{12}\mu)s|) + v_2^2(|\mu s|)) + \epsilon v(i\check{\rho}_2(u^{-1})/n, t) \\ &\geq (1 - 2\epsilon)(v(i\check{\rho}_2(u^{-1})/n, t) - v(s, t)) - (1 - 2\epsilon)^2(v_1^2(|(1+b_{12}\mu)s|) + v_2^2(|\mu s|))v(s, t) \\ &\quad + (1 - 3\epsilon)^2(v_1^2(|(1+b_{12}\mu)s|) + v_2^2(|\mu s|))v(i\check{\rho}_2(u^{-1})/n, t). \end{aligned}$$

Hence, combining the above with (94) and (95) for any $0 < \epsilon < 1/4$, we have for n, u sufficiently large,

$$1 + (1 - \epsilon)(v_1^2(|s+b_{12}t|) + v_2^2(|t|)) \geq [1 + (1 - 2\epsilon)(v_1^2(|(1+b_{12}\mu)s|) + v_2^2(|\mu s|))] [1 + (1 - 2\epsilon)v(s, t)]$$

$$\geq \left[1 + (1 - 3\epsilon) \left(v_1^2(|(1 + b_{12}\mu)s|) + v_2^2(|\mu s|)\right)\right] \left[1 + (1 - 3\epsilon)v(\sqrt[n]{p_2}(u^{-1})/n, t)\right],$$

holds for $|s| \in [\frac{i-1}{n}\sqrt[n]{p_2}(u^{-1}), \frac{i+2}{n}\sqrt[n]{p_2}(u^{-1})]$ with $x/2 \leq \frac{i}{n} \leq 2y$ and and $|t| \in [0, \sqrt[n]{v_2}(\ln u/u)]$, which completes the proof. \square

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KRZYSZTOF DĘBICKI, MATHEMATICAL INSTITUTE, UNIVERSITY OF WROCLAW, PL. GRUNWALDZKI 2/4, 50-384 WROCLAW, POLAND

E-mail address: `Krzysztof.Debicki@math.uni.wroc.pl`

ENKELEJD HASHORVA, DEPARTMENT OF ACTUARIAL SCIENCE, UNIVERSITY OF LAUSANNE, UNIL-DORIGNY 1015 LAUSANNE, SWITZERLAND

E-mail address: `enkelejd.hashorva.unil.ch`

PENG LIU, MATHEMATICAL INSTITUTE, UNIVERSITY OF WROCLAW, PL. GRUNWALDZKI 2/4, 50-384 WROCLAW, POLAND AND DEPARTMENT OF ACTUARIAL SCIENCE, UNIVERSITY OF LAUSANNE, UNIL-DORIGNY 1015 LAUSANNE, SWITZERLAND

E-mail address: `liupnankaimath@163.com`